Convex synthesis of symmetric modifications to linear systems

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Standard feedback control

$$\begin{aligned} \dot{x} &= A\,x + B\,u + d \\ z &= \begin{bmatrix} I \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u \end{aligned}$$

Objective: Minimize steady-state variance of z

$$\lim_{t \to \infty} \mathbf{E}\left(z^T(t)\, z(t)\right)$$

Design state feedback gain matrix $u = -Fx$

$$\dot{x} = \left(A - B\,F\right) x + d$$

Problem formulation

$$\dot{x} \;=\; \big(A \;-\; K(u)\big)\, x \;+\; d$$

- Control input $u \in \mathbb{R}^m$
- $K \in \mathbb{R}^{n \times n}$ linear function of u

Objective: Minimize steady-state variance of x

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Objective: Minimize steady-state variance of x

$$\lim_{t \to \infty} \mathbf{E}\left(x^T(t)\, x(t)\right)$$

- Constant u and symmetric $K(u)$

Related work

Design of system dynamics

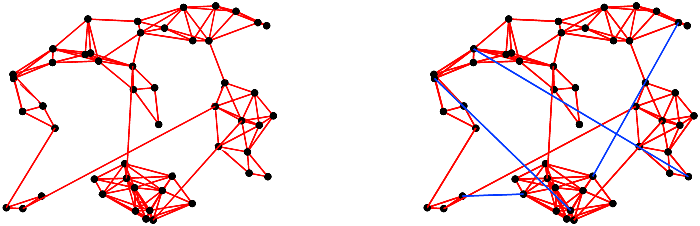
- Convex monotone systems (Rantzer and Bernhardsson CDC '14)
- Combination drug therapy (Jonsson, Rantzer, Matni, and Murray CDC '14)
- Vibrational control of bridges (Nelson, Rajamani, Gastineau, Schultz, and Wojtkiewicz '15)

Sparse feedback synthesis

- Via nonlinear programming (Lin, Fardad, and Jovanovic TAC '13)
- Quadratically invariant systems (Matni '15)
- Positive systems (Rantzer '12)

Examples – Growing consensus networks

$$\dot{x} \;=\; -\left(L \;+\; E\,\mathrm{diag}\{u\}\,E^T\right)\,x \;+\; d$$

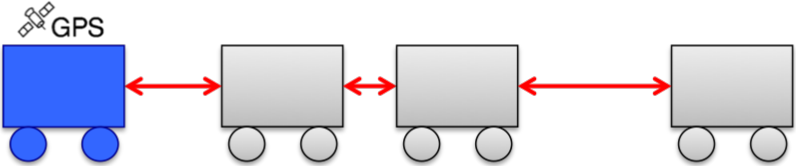


- Nodes x_i determine average via relative information exchange
- L is graph laplacian, E specifies edges, u specifies edge weights

Lin, Fardad, and Jovanovic, Allerton ‘12
Hassan-Moghaddam and Jovanovic, TCNS ‘15 (submitted);
also arXiv:1506.03437

Examples – Leader selection

$$\dot{x} \;=\; -\,(L \;+\; \mathrm{diag}\{u\})\;x \;+\; d$$

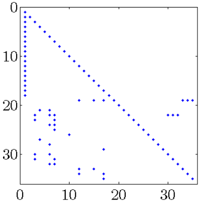


- Some nodes are 'leaders' with access to absolute measurements
- L is graph laplacian, u specifies leaders

Fitch and Leonard, CDC '13
Lin, Fardad, and Jovanovic, TAC '14

Examples – Combination drug therapy

$$\dot{x} = \left(A - \sum_{k=1}^{m} u_k D_k \right) x + d$$



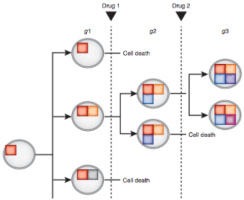


Image credit: Al-Lazikani et al '12

- Mutagen x_i mutates to x_j at rate A_{ji}
- Drug u_k kills x_i at rate $(D_k)_{ii}$

Rantzer and Bernhardsson CDC '14

Jonsson, Rantzer, Matni, and Murray CDC '14

Regularization

$$\text{minimize} \quad J(u) \quad + \quad u^T R\, u \quad + \quad \gamma \sum_i |u_i|$$

$\downarrow$ **variance amplification**

$\downarrow$ **control effort**

$\downarrow$ **sparsity-promoting term**

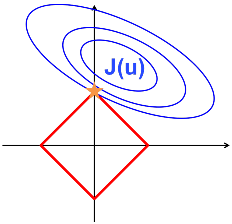
Regularization

$$\text{minimize} \quad J(u) \quad + \quad u^T R\, u \quad + \quad \gamma \sum_i |u_i|$$

$J(u)$ → **variance amplification**

$u^T R\, u$ → **control effort**

$\gamma \sum_i |u_i|$ → **sparsity-promoting term**



- γ specifies importance of sparsity
- $\gamma = 0$ yields dense u

Fardad, Lin, and Jovanovic ACC '11
Lin, Fardad, Jovanovic TAC '13

$\mathcal{H}_2$ norm of system

$$\dot{x} \;=\; \big(A \,-\, K(u)\big)\, x \,+\, d$$

$\mathcal{H}_2$ norm

$$J(u) \;:=\; \operatorname{trace}\left(X(u)\right)$$

$X(u)$ is state covariance

$$\big(A \,-\, K(u)\big)\, X(u) \,+\, X(u)\, \big(A \,-\, K(u)\big)^T \,+\, I \;=\; 0$$

In general, $J(u)$ nonconvex

$\mathcal{H}_2$ norm of symmetric system

$$\dot{x} \;=\; \big(A_s \;-\; K(u)\big)\,x \;+\; d$$

Lyapunov equation

$$\big(A_s \;-\; K(u)\big)X_s(u) \;+\; X_s(u)\big(A_s \;-\; K(u)\big)^T \;+\; I \;=\; 0$$

A_s and $K(u)$ symmetric $\implies$ explicit solution

$$J_s(u) \;:=\; \text{trace}\,\big(X_s(u)\big) \;=\; -\,\text{trace}\left(\frac{1}{2}\big(A_s \;-\; K(u)\big)^{-1}\right)$$

$\boxed{J_s(u)\ \text{convex}}$

SDP characterization

Regularized design problem

$$\begin{array}{ll} \text{minimize} & \text{trace}\left((K(u) - A_s)^{-1}\right) + u^T R u + \gamma \sum_i |u_i| \\ \text{subject to} & (K(u) - A_s) \succ 0 \end{array}$$

Schur complement

$$\begin{array}{ll} \text{minimize} & \text{trace}(Z) + u^T R u + \gamma \sum_i |u_i| \\ \text{subject to} & \begin{bmatrix} Z & I \\ I & K(u) - As \end{bmatrix} \succ 0 \end{array}$$

Key contributions

Central Idea:

Use symmetric component

$$A_s \;:=\; \frac{1}{2}\,(A \;+\; A^T)$$

to inform design for original system

- Convex characterization
- Relationship between A_s and A
  - Stability
  - Performance
  - Fidelity
- Customized Algorithm

Stability

$$A_s \text{ Hurwitz} \implies A \text{ Hurwitz}$$

Not a necessary condition

Implication

Designing u for A_s guarantees stability

$$\underbrace{\dot{x} = \big(A_s - K(u)\big)x + d}_{\text{stable}} \implies \underbrace{\dot{x} = \big(A - K(u)\big)x + d}_{\text{stable}}$$

Upper bound on $\mathcal{H}_2$ norm

Symmetric system's $\mathcal{H}_2$ norm upper bounds original

$$\underbrace{\dot{x} \;=\; A\,x \;+\; d}_{\|\cdot\|_2} \quad\leq\quad \underbrace{\dot{x} \;=\; A_s\,x \;+\; d}_{\|\cdot\|_2}$$

Convex upper bound on performance

Implication

A controller u designed for A_s will perform better with A

$$\underbrace{\dot{x} \;=\; \big(A \;-\; K(u)\big)\,x \;+\; d}_{J(u)} \quad\leq\quad \underbrace{\dot{x} \;=\; \big(A_s \;-\; K(u)\big)\,x \;+\; d}_{J_s(u)}$$

Small perturbations

- Perturb system dynamics of

$$\dot{x} \ = \ A_n \, x \ + \ d$$

by antisymmetric matrix A_a

$$A \ = \ A_n \ + \ \epsilon \, A_a$$

- First order correction to $\mathcal{H}_2$ norm is 0

Implication

A_s is a very good approximation when A is close to normal

$$J_s(u) \ = \ J(u) \ + \ O(\epsilon^2)$$

Design procedure

- Form $A_s = \frac{1}{2}(A + A^T)$

- Solve convex symmetric design problem

$$u^* \;=\; \operatorname{argmin} \;\; \mathrm{trace}\,(Z) \,+\, u^T R\, u \,+\, \gamma \sum_i |u_i|$$

$$\text{subject to} \;\; \begin{bmatrix} Z & I \\ I & K(u) - A_s \end{bmatrix} \succ 0$$

- Implement u^* on original system A

Alternating Direction Method of Multipliers (ADMM)

- Splitting method
- Auxiliary variable V

$$\begin{array}{ll} \underset{V,u}{\text{minimize}} & \text{trace}\,(V^{-1}) \; + \; u^T R\, u \; + \; \gamma \sum_i |u_i| \\ \text{subject to} & V \; + \; A_s \; - \; K(u) \; = \; 0 \\ & V \; \succ \; 0 \end{array}$$

- Form augmented Lagrangian

$$\begin{aligned} \mathcal{L}_\rho(V, u, Y) \;\; := \;\; & \text{trace}\,(V^{-1}) \; + \; u^T R\, u \; + \; \gamma \sum_i |u_i| \\ & + \; \langle Y, V \; + \; A_s \; - \; K(u) \rangle \; + \; \frac{\rho}{2} \, \| V \; + \; A_s \; - \; K(u) \|_F^2 \end{aligned}$$

- Update variables **separately**

$$V_{k+1} = \underset{V}{\operatorname{argmin}} \ \mathcal{L}_\rho(V, u_k, Y_k)$$

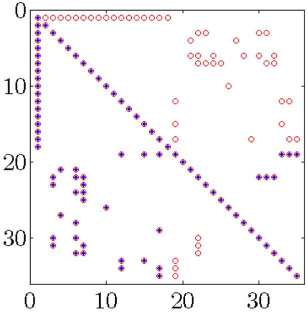
$$u_{k+1} = \underset{u}{\operatorname{argmin}} \ \mathcal{L}_\rho(V_{k+1}, u, Y_k)$$

$$Y_{k+1} = Y_k + \rho\,(V_{k+1} + A_s - K(u_{k+1}))$$

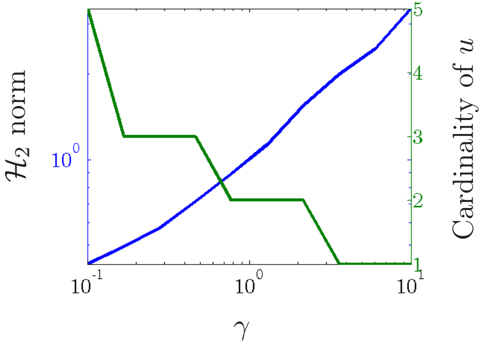
Boyd, Parikh, Chu, Peleato, Eckstein ‘11

HIV combination drug therapy

$$\dot{x} = \left(A - \sum_{k=1}^{m} u_k D_k \right) x + d$$



Sparsity patterns of A ($*$) and A_s ($\circ$)



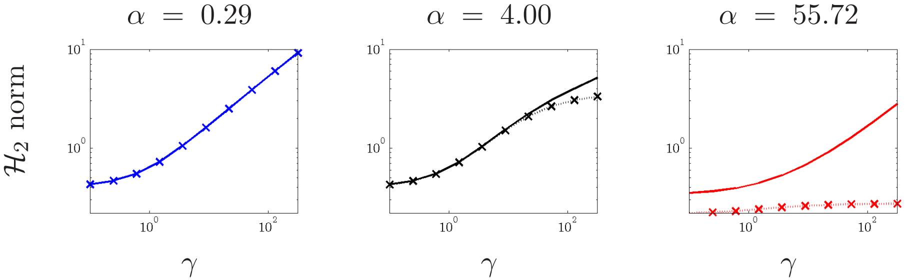
Model credit: Klein et al '12

HIV combination drug therapy-inspired example

Artificially inflate off-diagonal entries

$$A_\alpha = (I \circ A) + \alpha \frac{\|I \circ A\|_{\max}}{\|A - (I \circ A)\|_{\max}} (A - I \circ A)$$

Closed loop perf. of original (—) and symmetric (- -×- -) systems



Conclusions

Contributions

- Convex characterization for symmetric systems
- Design procedure
  - Stability guarantee
  - Upper bound on performance
  - Good approximation when 'mostly' normal
- Customized, scalable algorithm

Ongoing work

- Nonlinear programming for asymmetric systems
- Time-varying u

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