Identification of spatially-localized flow structures via sparse POD

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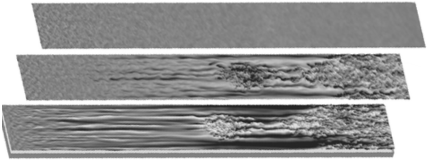
Proper Orthogonal Decomposition



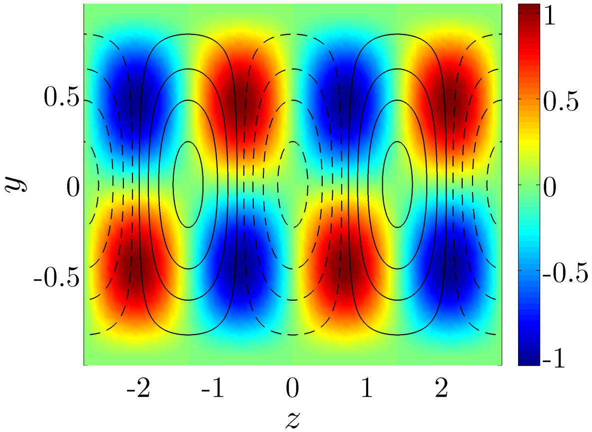
POD used to extract important flow structures

- Experimental measurements
- CFD data
- **Linearized dynamics**

Sirovich '87, Berkooz et al '03, Lumley '07, Schmid '07

Standard POD

Periodic flow structures - infinite spatial support



Want *localized* flow structures - *sparse* spatial support

Foures, Caulfield, and Schmid '13

Sparse POD

Proper Orthogonal Decomposition

$$\begin{aligned} \text{maximize} \quad & x^* \Sigma x \\ \text{subject to} \quad & x^* x = 1 \end{aligned}$$

- (dense) principal eigenvector

Sparse POD

Sparse Proper Orthogonal Decomposition

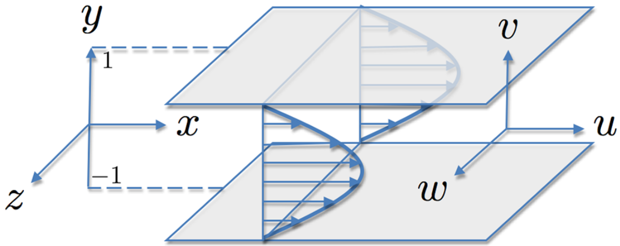
$$\begin{aligned} \text{maximize} \quad & x^* \Sigma x - \gamma \sum_i |x_i| \\ \text{subject to} \quad & x^* x = 1 \end{aligned}$$

- ~~(dense) principal eigenvector~~
- ℓ_1-norm promotes sparsity
- $\gamma > 0$ specifies relative *importance* of sparsity

Zou et al '06, d'Aspremont et al '07

Fluids Applications

In many applications, sparsity desired in *different coordinates*



Channel flow dynamics described by x but want sparse $\mathcal{C}x$

$$x := \begin{bmatrix} v \\ \eta \end{bmatrix} \qquad \mathcal{C}x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Sparse POD: Convex Formulation

$$\begin{aligned} \text{maximize} \quad & x^* \Sigma x \; - \; \gamma \sum_i \left| (\mathcal{C}\, x)_i \right| \\ \text{subject to} \quad & x^* x = 1 \end{aligned}$$

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- Introduce $X = xx^*$

$$\begin{aligned} \text{maximize} \quad & \text{trace}(\Sigma\, X) - \sum_{i,j} \left| (\mathcal{C} X \mathcal{C}^*)_{ij} \right| \\ \text{subject to} \quad & \text{trace}(X) = 1 \\ & X \succeq 0 \\ & \text{rank}(X) = 1 \end{aligned}$$

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$\mathcal{C} = I$, Zou et al ‘06, d’Aspremont et al ‘07, Ma ‘11

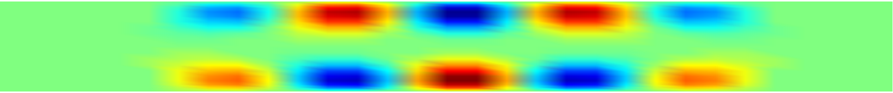
Alternating Direction Method of Multipliers

$$
\begin{array}{ll}
\text{maximize} & \mathrm{trace}(\Sigma X) \; - \; \gamma \sum_{i,j} | Z_{ij} | \\
\text{subject to} & \mathcal{C} X \mathcal{C}^* - Z = 0 \\
& X \succeq 0, \quad \mathrm{trace}(X) = 1
\end{array}
$$

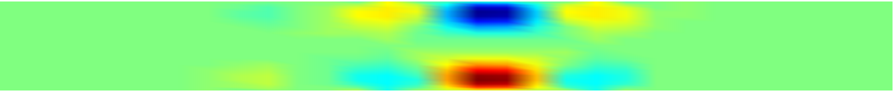
Solve by alternating between maximization over X and over Z

Boyd et al ‘11

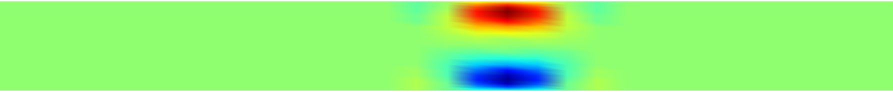
Sparse Transient Growth Analysis



(a) $\gamma = 0.1150$



(b) $\gamma = 0.9326$



(c) $\gamma = 10$

Figure: Sparse initial spanwise velocity for a streamwise constant channel flow.

Conclusion

New sparse POD formulation

- Promote **sparsity in an image** of the optimization variable
- More general notion of sparsity

ADMM makes problem tractable

- Computationally efficient algorithm
- Application to channel flow

Future

- Use of reduced order representations in sparse POD
- Application to more complicated flows

Augmented Lagrangian

$$\mathcal{L}_\rho(X, Z, \Lambda) = -\text{trace}(\Sigma X) + I_{\mathcal{A}}(X) + \gamma \|Z\|_{\ell_1} - \text{trace}(\Lambda^T(\mathcal{C}X\mathcal{C}^* - Z)) + \frac{\rho}{2}\|\mathcal{C}X\mathcal{C}^* -$$

Solve with sequence

Boyd et al '11

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$\mathbf{X^{k+1}} = \text{argmin}\ L_\rho(X, Z^k, \Lambda^k)$

Project eigenvalues onto simplex, $\mathcal{O}(n^3)$

Boyd et al '11

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Soft-thresholding operator, $\mathcal{O}(m^2)$

Boyd et al '11

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Soft-thresholding operator, $\mathcal{O}(m^2)$

$$\Lambda^{k+1} = \Lambda^k - \rho(\Pi X^{k+1}\Pi^T - Z^{k+1})$$

Boyd et al ’11