An ADMM algorithm for optimal sensor and actuator selection

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Motivation

- Systematic approach to selecting sensors and actuators
  - Phasor Measurement Units in power networks
  - Autonomous formations of vehicles
  - Flexible wing aircraft

Recent work

Particular measurement models

- Linear measurements (Joshi, Boyd '09)
- Cramer-Rao lower bound (Roy, Chepuri, Leus '13)
- Optimal experiment design (Kekatos, Giannakis, Wollenberg '12)

Related approaches

- Maximize effect of actuators (Summers, Lygeros '14)
- ADMM for nonconvex formulation (Masazade, Fardad, Varshney '12)

Convex characterization as Semidefinite Program

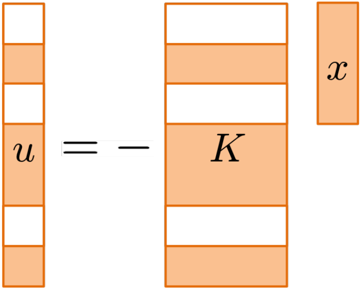
- SDP formulation (Polyak, Khlebnikov, Shcherbakov '13)
- Discrete time (Munz, Pfister, Wolfrum '14)

Problem formulation

LTI system with **many potential actuators**

$$
\begin{aligned}
\dot{x} &= Ax + B_1 d + B_2\, u \\
z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u
\end{aligned}
$$

Objective: Minimize closed loop $\mathcal{H}_2$ norm using a **few actuators**



Actuator selection via regularization

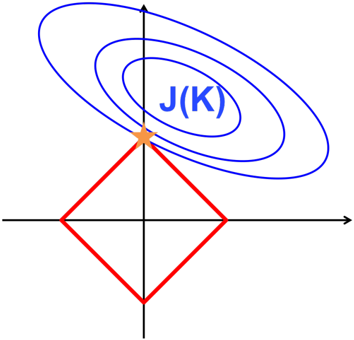
$$\text{minimize} \quad \underbrace{J(K)}_{\substack{\textbf{variance amplification} \\ \textbf{(closed loop } \mathcal{H}_2 \textbf{ norm)}}} \quad + \quad \underbrace{\gamma \sum_i \|\mathrm{e}_i^T K\|_2}_{\substack{\textbf{row-sparsity-promoting} \\ \textbf{penalty function}}}$$

Actuator selection via regularization

$$\text{minimize} \quad J(K) \quad + \quad \gamma \sum_i \|e_i^T K\|_2$$

$\downarrow$ **variance amplification** **(closed loop $\mathcal{H}_2$ norm)**

$\downarrow$ **row-sparsity-promoting penalty function**



- γ specifies relative importance of sparsity
- $\gamma = 0$ uses all actuators

Lin, Fardad, Jovanovic, ACC '11, TAC '13

State feedback $\mathcal{H}_2$ problem

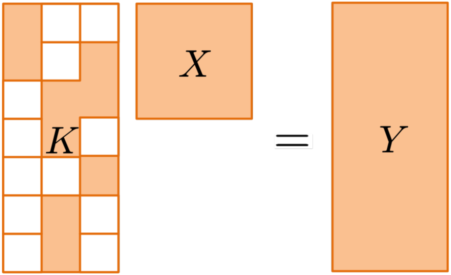
$$\begin{array}{ll} \underset{X,K}{\text{minimize}} & \operatorname{trace}\left(Q\,X\right) \;+\; \operatorname{trace}\left(R\,K\,X\,K^T\right) \\[1em] \text{subject to} & (A \;-\; B_2\,K)\,X \;+\; X\,(A \;-\; B_2\,K)^T \;+\; B_1\,B_1^T \;=\; 0 \\[1em] & X \;\succ\; 0 \end{array}$$

LMI formulation

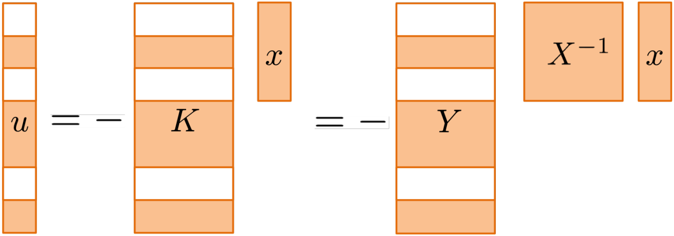
Standard change of variables $Y := KX$ yields convex form

$$\begin{array}{ll} \underset{X,Y}{\text{minimize}} & \operatorname{trace}\left(Q\,X\right) \;+\; \operatorname{trace}(R\,Y\,X^{-1}\,Y^T) \\[2ex] \text{subject to} & A\,X \;-\; B_2\,Y \;+\; X\,A^T \;-\; Y^T\,B_2^T \;+\; B_1\,B_1^T \;=\; 0 \\[2ex] & X \;\succ\; 0 \end{array}$$

- Challenge: impose structure on K in (X, Y) coordinates
  - Linear constraints on K become *nonlinear* on Y and X



- Row-sparsity is preserved



Actuator selection - Semidefinite Program

$$\underset{X,Y}{\text{minimize}} \quad \underbrace{\text{trace}(Q\,X) \;+\; \text{trace}(R\,Y\,X^{-1}\,Y^T) \;+\; \gamma \sum \|\mathrm{e}_i^T {\color{red}Y}\|_2}_{f(X,Y)}$$

$$\text{subject to} \quad A\,X \,-\, B_2\,Y \,+\, X\,A^T \,-\, Y^T\,B_2^T \,+\, B_1\,B_1^T \,=\, 0$$

$$X \succ 0$$

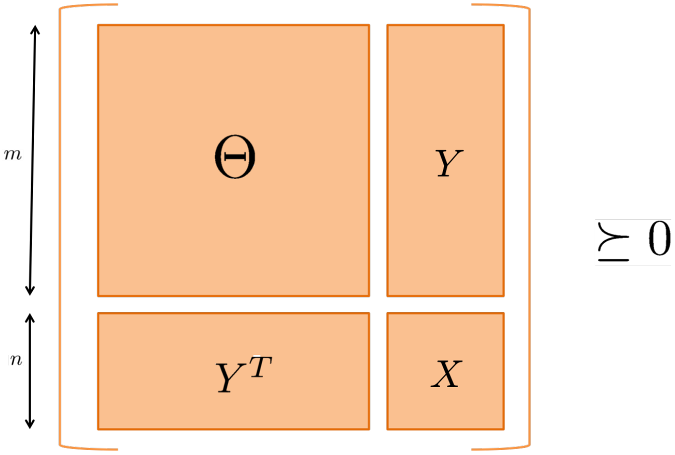
- Promote row sparsity of ${\color{red}Y}$ instead of ${\color{red}K}$

Polyak, Khlebnikov, Shcherbakov, ECC '13

Munz, Pfister, Wolfrum, TAC '14

Computational complexity

$$\mathrm{trace}(\,R\,Y\,X^{-1}\,Y^T\,) \;=\; \mathrm{trace}(R\,\Theta)$$



Worst case computational complexity: $\mathcal{O}\left((n+m)^6\right)$

Alternating Direction Method of Multipliers (ADMM)

- **Splitting method** for convex functions with linear constraints

Form **augmented Lagrangian**

$$\mathcal{L}_\rho(X, Y, \Lambda) \;\; := \;\; f(X, Y) \; + \; \langle \Lambda, h(X, Y) \rangle \; + \; \frac{\rho}{2} \, \|h(X, Y)\|_F^2$$

$$h(X, Y) \;\; := \;\; A\, X \; - \; B_2\, Y \; + \; X\, A^T \; - \; Y^T\, B_2^T \; + \; B_1\, B_1^T$$

- Update variables **separately**

$$
\begin{aligned}
X_{k+1} &= \arg\min_{X} \; \mathcal{L}_\rho(X,\, Y_k,\, \Lambda_k) \\
Y_{k+1} &= \arg\min_{Y} \; \mathcal{L}_\rho(X_{k+1},\, Y,\, \Lambda_k) \\
\Lambda_{k+1} &= \Lambda_k \;+\; \rho\, h(X_{k+1},\, Y_{k+1})
\end{aligned}
$$

Boyd, Parikh, Chu, Peleato, Eckstein‘11

Y-minimization

$$\underset{Y}{\text{minimize}} \;\; \gamma \sum \|\mathrm{e}_i^T Y\|_2 \; + \; \frac{\rho}{2} \|B_2\, Y \; + \; Y^T B_2^T \; + \; V_k\|_F^2$$

- **Group LASSO**

Y-minimization

$$\underset{Y}{\text{minimize}} \;\; \gamma \sum \|e_i^T Y\|_2 \; + \; \frac{\rho}{2} \|B_2\, Y \; + \; Y^T B_2^T \; + \; V_k\|_F^2$$

- **Group LASSO**

- Proximal method: sequentially approximate with

$$\underset{Y}{\text{minimize}} \;\; \gamma \sum \|e_i^T Y\|_2 \; + \; \frac{1}{2} \|Y \; - \; \bar{V}_k\|_F^2$$

- Computational complexity of $\mathcal{O}(n^2 m)$

X-minimization

$$\underset{X}{\text{minimize}} \;\; \text{trace}\left(X\,Q \; + \; X^{-1}\,Y_k^T\,R\,Y\right) \; + \; \frac{\rho}{2}\,\|A\,X \; + \; X\,A^T \; + \; U_k\|_F^2$$

- Can formulate as SDP (worst case computational complexity $\mathcal{O}(n^6)$)

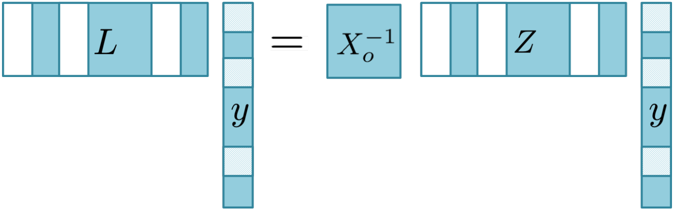
X-minimization

$$\underset{X}{\text{minimize}} \;\; \text{trace}\left(X\,Q \;+\; X^{-1}\,Y_k^T\,R\,Y\right) \;+\; \frac{\rho}{2}\,\|A\,X \;+\; X\,A^T \;+\; U_k\|_F^2$$

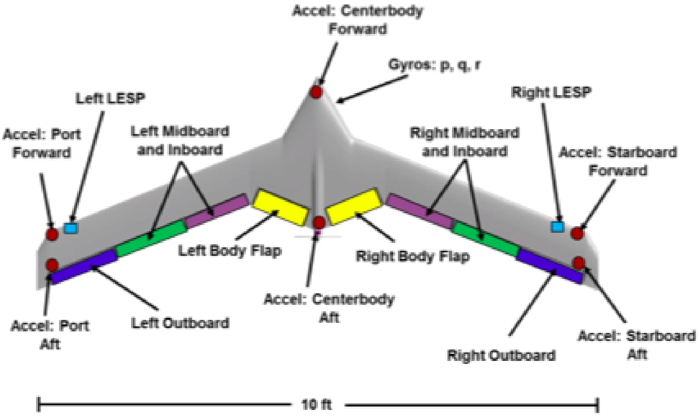
- Can formulate as SDP (worst case computational complexity $\mathcal{O}(n^6)$)
- Projected Newton's method
  - Find search directions with conjugate gradient (n^5)
  - Project onto $\{X : X \succ 0\}$

Sensor Selection: SDP Formulation

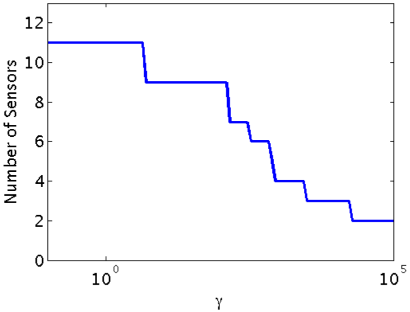
Change of coordinates $Z := X_o L$

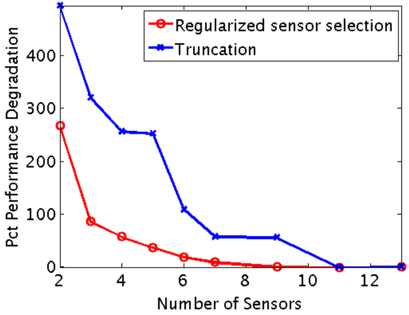


Example - Flexible Wing Aircraft



- Select subset of sensors to minimize estimation error





- Using less than half the sensors degrades performance by only $\sim 20\%$

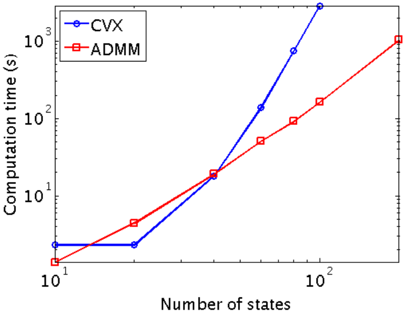
Example - Mass Spring System

- Dynamics

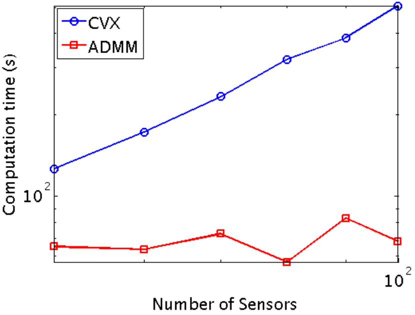
$$\ddot{p}_i = (p_{i+1} - p_i) + (p_{i-1} - p_i) + d_i$$

- Sensors
  - Position of each mass
  - Velocity of each mass

Computation time for $\gamma = 100$



n states and n outputs



50 states and m outputs

Conclusions

- Convex characterization of sensor or actuator selection
- Splitting algorithm
  - Solve simpler sub-problems
  - Scaling dependent on number of states

Future work

- More efficient X-minimization
- Customized Interior Point Method
- Alternative splitting methods

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Collaboration on aircraft example

- Marty Brenner, NASA
- Claudia Moreno and Harald Pfifer

Dual Problem

$$g(Y) = \underset{e_i^T U \le \gamma}{\text{maximize}} \langle U, Y \rangle$$

Lagrangian

$$\begin{aligned} L(X, Y, \Lambda, M) \quad = \quad & \operatorname{trace}(QX + X^{-1} Y^T R Y) + \langle U, Y \rangle \\ & + \left\langle \Lambda, AX + XA^T - BY - Y^T B^T + V \right\rangle - \langle M, X \rangle \end{aligned}$$

where $M \succeq 0$

$$\begin{bmatrix} \nabla_X L \\ \nabla_Y L \end{bmatrix} = \begin{bmatrix} Q - X^{-1} Y^T R Y X^{-1} + A^T \Lambda + \Lambda A \\ 2RYX^{-1} + U - 2B_2^T \Lambda \end{bmatrix}$$

Dual Problem

$$\begin{array}{ll} \text{maximize} & \langle V, \Lambda \rangle \\ \text{subject to} & Q + \Lambda A + A^T \Lambda - (B_2^T \Lambda - \frac{1}{2} U)^T R^{-1} (B_2^T \Lambda - \frac{1}{2} U) \succeq 0 \\ & \|e_i^T U\|_2 \le \gamma \end{array}$$

Sensor Selection

$$\begin{aligned} \dot{x} &= A\,x + B_1\,d \\ y &= C\,x + \eta \end{aligned}$$

- Observer estimates state from output

$$\dot{\hat{x}} = A\,\hat{x} + L\,(y - \hat{y})$$

Sensor Selection

$$\begin{aligned} \dot{x} &= A\,x + B_1\,d \\ y &= C\,x + \eta \end{aligned}$$

- Observer estimates state from output

$$\dot{\hat{x}} = A\,\hat{x} + L\,(y - \hat{y})$$

- Steady-state variance of the estimation error $x - \hat{x}$

$$J(L) = \operatorname{trace}\left(X_o\,B_1\,B_1^T + X_o\,L\,L^T\right)$$

- Observability gramian $X_o \succ 0$

$$(A - L\,C)^T X_o + X_o\,(A - L\,C) + I = 0.$$

Sensor Selection: SDP Formulation

Change of coordinates $Z \; := \; X_o L$

$$\underset{X_o,\,Z}{\text{minimize}} \quad \operatorname{trace}\left(X_o\, B_1\, B_1^T \;+\; X_o^{-1}\, Z\, Z^T\right) \;+\; \gamma \sum_{i=1}^{r} \|Z\, \mathrm{e}_i\|_2$$

$$\text{subject to} \quad A^T X_o \;+\; X_o A \;-\; C^T Z \;-\; Z C \;+\; I \;=\; 0$$

$$X_o \;\succ\; 0$$