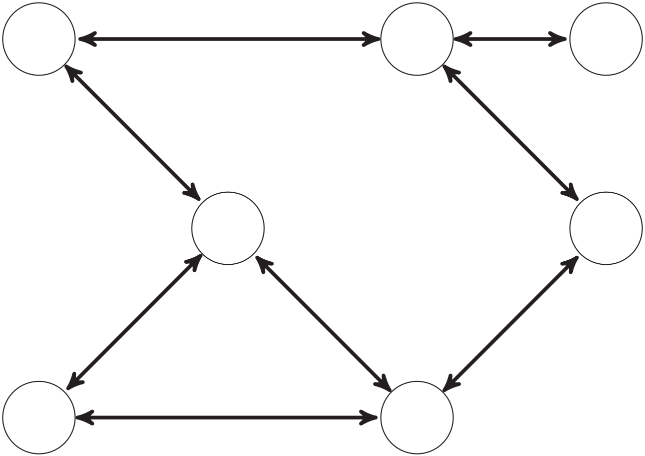
Leader selection in directed consensus networks

Neil K. Dhingra, Marcello Colombino
and Mihailo R. Jovanović

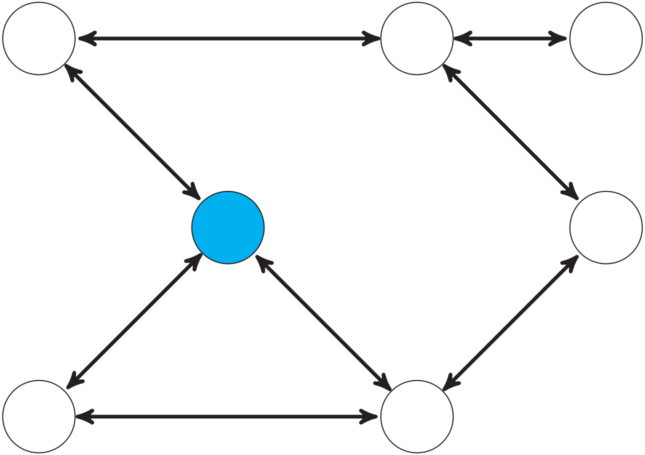
2016 IEEE Conference on Decision and Control, Las Vegas, NV

Consensus networks



- Relative information exchange
- Distributed averaging

Leader selection



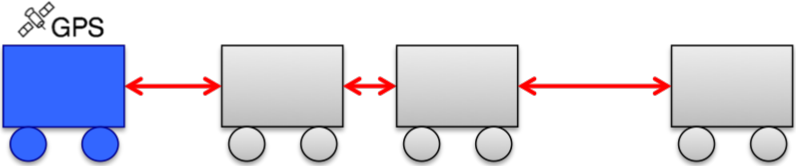
- Some nodes have *absolute* information

Patterson and Bamieh, TAC '10

Lin, Fardad, and Jovanović, CDC '11, TAC '14

Leader selection in consensus networks

$$\dot{x} = -(L + \text{diag}(u))\, x + d$$

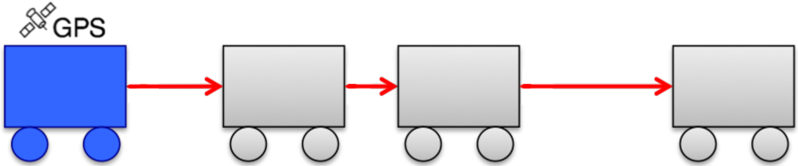


- 'Noise-free' leader i – $u_i \to \infty$
- 'Noise-corrupted' leader i – u_i finite

Patterson and Bamieh, TAC '10
Lin, Fardad, and Jovanović, CDC '11, TAC '14

Next questions

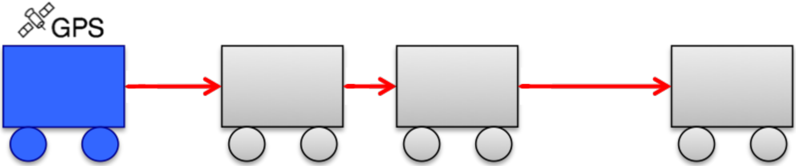
Directed networks – asymmetric information exchange



Patterson, McGlohon, Dyagilev TCNS '16
Lin '16, arXiv:1606.02269

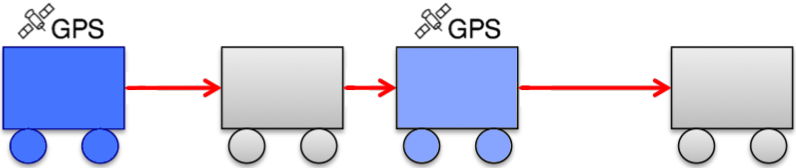
Next questions

Directed networks – asymmetric information exchange



Patterson, McGlohon, Dyagilev TCNS ‘16
Lin ‘16, arXiv:1606.02269

Design vector of leader weights – nonhomogeneous noise corruption



Noise-corrupted leader selection problems

Given: Weakly connected directed network with weighted edges

Problem 1: Select set $\mathcal{N}$ of k leaders with given weight κ

$$u = \sum_{i \in \mathcal{N}} \kappa \, \mathrm{e}_i$$

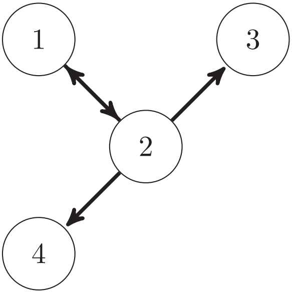
Problem 2: Select set $\mathcal{N}$ of k leaders *and* design leader weights

$$u = \sum_{i \in \mathcal{N}} \kappa_i \, \mathrm{e}_i$$

- Penalize u in terms of budget, etc.

Stability for leader selection

- Directed network → not all leaders stabilizing



- Leaders u stablizing $\iff u \circ w \neq 0 \;\; \forall \;\; w \,|\, w^T L = 0$ (Thm 4)
  - Path from set of leaders to every node
  - Strongly connected → any leader is stabilizing
  - Independent of leader weight scaling

Problem Formulation

$$\begin{aligned}\dot{x} &= -(L + \mathrm{diag}(u))\, x + B\, d \\ z &= Cx\end{aligned}$$

Objective: Select leaders u to minimize

$\mathcal{H}_2$: minimize variance amplification

$$\lim_{t \to \infty} \mathbf{E}\left(z^T(t)\, z(t)\right)$$

$\mathcal{H}_\infty$: minimize worst-case amplification

$$\sup_{\|d\|=1} \frac{\|z\|}{\|d\|}$$

Undirected networks – symmetry

$B = C = I \to \mathcal{H}_2$ norm is trace(X)

$$-\,(L \,+\, \text{diag}(u))\,X \,-\, X\,(L \,+\, \text{diag}(u))^T \,+\, I \,=\, 0$$

Undirected networks – symmetry

$B = C = I \rightarrow \mathcal{H}_2$ norm is trace(X)

$$-\,(L \,+\, \mathrm{diag}(u))\,X \,-\, X\,(L \,+\, \mathrm{diag}(u))^T \,+\, I \,=\, 0$$

Undirected network $\rightarrow L$ symmetric $\rightarrow X \,=\, \dfrac{1}{2}\,(L \,+\, \mathrm{diag}(u))^{-1}$

Undirected networks – symmetry

$B = C = I \rightarrow \mathcal{H}_2$ norm is trace(X)

$$-\left(L + \text{diag}(u)\right) X - X \left(L + \text{diag}(u)\right)^T + I = 0$$

Undirected network $\rightarrow$ L symmetric $\rightarrow$ $X = \dfrac{1}{2}\left(L + \text{diag}(u)\right)^{-1}$

- Rank one update rules for inverse $\rightarrow$ explicit expression

  Fitch and Leonard, CDC '13, TAC '16

- Schur complement and convex relaxation of cardinality

  Lin, Fardad, and Jovanovic, CDC '11, TAC '14

Directed networks – positivity

$$\begin{aligned} \dot{x} &= A\,x + B\,d \\ z &= Cx \end{aligned}$$

- Positive system, $x(0) \geq 0,\ d(t) \geq 0 \rightarrow x(t) \geq 0 \ \forall\, t$
  - $B, C \geq 0$
  - A Metzler (off-diagonal elements nonnegative)
- **Directed** consensus networks with leaders
  - $-L$ is Metzler
  - $A_{\rm cl} := -(L + \mathrm{diag}(u)) \rightarrow$ adding leaders always preserves positivity

Decentralized control of positive systems

Decentralized control of positive systems

- SDP formulation (Tanaka and Langbort, TAC '11)
- LP formulation (Rantzer '12, '15)

Structured decentralized control

- Combination drug therapy (Jonsson et al CDC '14)
- $\mathcal{L}_1$ optimal control (Rantzer and Bernhardsson CDC '14)
  (Colaneri et al '14)
- $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control (Dhingra et al ECC '16)
- Robust control (Colombino et al CDC '16)
- Time-varying control (Dhingra et al SCL '16, submitted)

Gradient of $\mathcal{H}_2$ norm

$$\nabla_{u_i} \|\cdot\|_{\mathcal{H}_2} \;=\; -2\,(X_c\,X_o)_{ii}$$

controllability and observablity gramians

$$0 \;=\; A_{\rm cl}\,X_c \,+\, X_c\,A_{\rm cl}^T \,+\, BB^T$$
$$0 \;=\; A_{\rm cl}^T\,X_o \,+\, X_o\,A_{\rm cl} \,+\, C^T C$$

- $X_c X_o$ used in balanced truncation
- X_c, X_o nonnegative $\to$ $\mathcal{H}_2$ norm monotone in leaders

Dhingra, Colombino, and Jovanović ECC '16

Subgradient map of $\mathcal{H}_\infty$ norm

$$\|\cdot\|_{\mathcal{H}_\infty} \;=\; \sup_{\omega}\; \bar{\sigma}\left(C\,(j\omega I - A_{\rm cl})^{-1}\,B\right) \;=\; \bar{\sigma}\left(C A_{\rm cl}^{-1} B\right) \;=\; w_j^T (C A_{\rm cl}^{-1} B) v_j$$

$$\partial_{u_i} \|\cdot\|_{\mathcal{H}_\infty} \;=\; -\sum_j \alpha_j \; \left(\mathrm{e}_i^T A_{\rm cl}^{-1} B\, v_j\right) \left(w_j^T\; C A_{\rm cl}^{-1} \mathrm{e}_i\right)$$

convex combination ($\alpha_j \in [0,1]$, $\sum \alpha_j = 1$)

- v_j, w_j left/right possibly nonunique principal singular vectors
- monotone in leaders
- Strongly connected $\rightarrow$ continuously differentiable

Dhingra, Colombino, and Jovanović ECC ‘16

Leader selection problem strategies

Problem 2: Select k leaders *and* design leader weights

Select leaders

$$u_\lambda \;=\; \text{minimize} \;\; \|\cdot\|_{\mathcal{H}_2/\mathcal{H}_\infty} \;+\; \|u\|_2^2 \;+\; \lambda\|u\|_1$$

- Increase λ until $\text{card}(u_\gamma) \;=\; k$

Design leader weights

$$\begin{array}{ll} \text{minimize} & \|\cdot\|_{\mathcal{H}_2/\mathcal{H}_\infty} \;+\; \|u\|_2^2 \\ \text{subject to} & \text{sp}(u) \;\in\; \text{sp}(u_\gamma) \end{array}$$

Leader selection

Problem 1: Select k leaders with given weight κ

- Bounds
  - **Upper bound:** Greedy algorithm
  - **Lower bound:** (enabled by convexity)

$$\begin{array}{rl} \text{minimize} & \|\cdot\|_{\mathcal{H}_2/\mathcal{H}_\infty} \\ \text{subject to} & \mathbb{1}^T u \;\leq\; k\kappa \\ & u_i \;\in\; [0,\,1] \end{array}$$

Leader selection

Problem 1: Select k leaders with given weight κ

- Bounds
  - **Upper bound:** Greedy algorithm
  - **Lower bound:** (enabled by convexity)

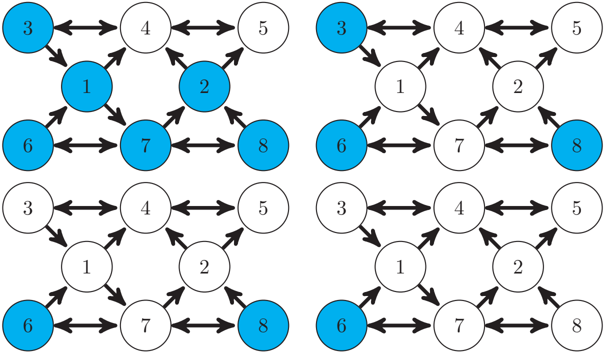
$$\begin{array}{ll} \text{minimize} & \|\cdot\|_{\mathcal{H}_2/\mathcal{H}_\infty} \\ \text{subject to} & \mathbb{1}^T u \;\leq\; k\kappa \\ & u_i \;\in\; [0,\, 1] \end{array}$$

- Balanced networks
  - L^T Laplacian $\to \frac{1}{2}(L + L^T)$ Laplacian
  - Better upper bound using best undirected leaders
  - Undirected leaders will have better performance on directed graph

Lin, Fardad, and Jovanovic, CDC '11, TAC '14
Fitch and Leonard CDC '13, TAC '16
Dhingra and Jovanović, ACC '15

Leader selection in directed networks

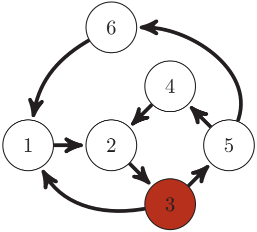
ℓ_1 regularization with varying emphasis on sparsity



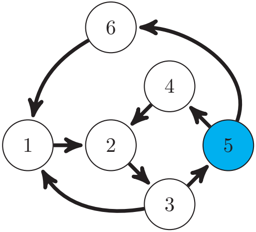
$\mathcal{H}_2$ leader selection for $N = 6, 3, 2, 1$ ($\lambda = 0.65, 1.97, 3.43, 125.9$)

Leader selection in directed networks

ℓ_1 regularization



$\mathcal{H}_2$-optimal leader



$\mathcal{H}_\infty$-optimal leader

Acknowledgements

Support:

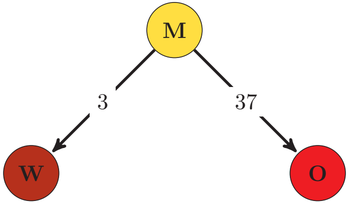
- NSF ECCS-1407958
- University of Minnesota DDF
- UMN Informatics Institute Transdisciplinary Faculty Fellowship
- Swiss National Science Foundation 2–773337–12

Additional thanks:

- Useful discussions – Katie Fitch, Princeton University/Technische Universität München

Leaders as index of importance

- Proxy for identifying influential/important nodes
- Sports example: edges from victor to loser weighted by score



M 17 – 14 W
O 14 – 51 M

Leaders as index of importance

2015 – 2016 College Football Season

λ	0.225	0.6	1	10
Teams	8	6	4	2
AP #1	Alabama	Alabama	Alabama	Alabama
AP #4	Ohio State	Ohio State	Ohio State	Ohio State
AP #2	Clemson	Clemson	Clemson	
AP #8	Houston	Houston	Houston	
AP #3	Stanford	Stanford		
AP NR	S. Illinois	S. Illinois		
AP #12	Michigan			
AP #10	Mississippi			

Table: Leaders selected for different values of λ.