On the Optimal Control Problem for a Class of Monotone Bilinear Systems

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Positive systems

$$\begin{aligned} \dot{x} &= A\,x + B\,d \\ z &= Cx \end{aligned}$$

- Positive system, $x(0) \geq 0,\ d(t) \geq 0 \rightarrow x(t) \geq 0 \ \forall t$
- Linear systems
  - $B, C \geq 0$
  - A Metzler (off-diagonal elements nonnegative)
- E.g., Population/resource dynamics, Markov models

Structured decentralized control

$$\begin{aligned} \dot{x} &= (A + K(u))\, x + B\, d \\ z &= \begin{bmatrix} Q^{1/2} x \\ R^{1/2} u \end{bmatrix} \end{aligned}$$

- $K(u)$ diagonal – preserves positivity

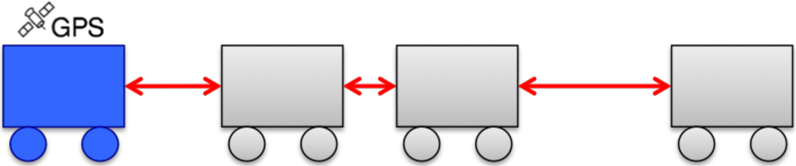
Objective: Design u to minimize worst-case amplification

$$\sup_{\|d\|_{\text{pow}}^2 \le 1} \frac{\|z\|_{\text{pow}}^2}{\|d\|_{\text{pow}}^2}$$

$\mathcal{H}_\infty$ norm of system when u constant

Leader selection in directed consensus networks

$$\dot{x} \;=\; -\,(L \,+\, \mathrm{diag}\{u\})\,x \,+\, d$$

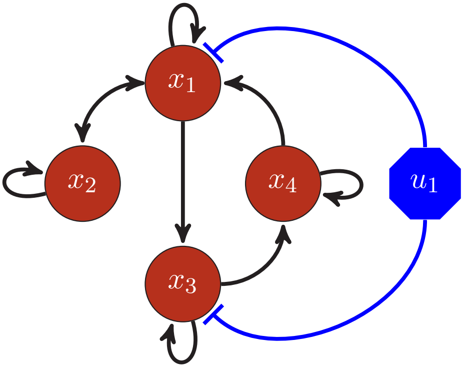


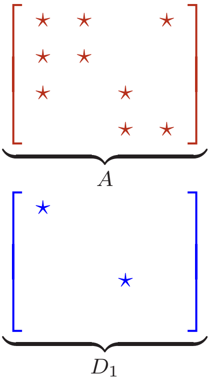
- Nodes x_i determine average via *relative* information exchange
- *Some* nodes are 'leaders' with access to *absolute* measurements
- L is directed graph laplacian, u specifies leaders

Fitch and Leonard, CDC '13, TAC '16
Lin, Fardad, and Jovanovic, TAC '14

Combination drug therapy

$$\dot{x} = \left(A + \sum_{k=1}^{m} u_k D_k \right) x + d$$





- Mutant x_i mutates to x_j at rate A_{ji}
- Drug u_k kills x_i at rate $(D_k)_{ii}$

Rantzer and Bernhardsson CDC '14
Jonsson, Rantzer, Matni, and Murray CDC '14

Decentralized control of positive systems

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- SDP formulation (Tanaka and Langbort, TAC '11)
- LP formulation (Rantzer '12, '15)

Structured decentralized control

- Combination drug therapy (Jonsson et al CDC '14)
- $\mathcal{L}_1$ control (Rantzer and Bernhardsson CDC '14)

  (Colaneri et al '14)
- $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control (Dhingra et al ECC '16)

Beyond static structured feedback control

$$
\begin{aligned}
\dot{x} &= (A + K(u))\,x + B\,d \\
z &= \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix}
\end{aligned}
$$

- Time-varying $u(t)$
- Feedback law $u(x)$

Question: Can we achieve better performance with $u(t)$ or $u(x)$?

Main results

Constant control $u(t) \equiv \bar{u}$ optimal

- Performance metric: worst case induced power

$$\sup_{\|d\|_{\text{pow}}^2 \leq 1} \frac{\|z\|_{\text{pow}}^2}{\|d\|_{\text{pow}}^2}$$

- Optimal over closure of periodic functions

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Uncertain model A and uncertain control u

- assume largest A_{ij}
- assume smallest $K(u)$

Notation

Power semi-norm

$$\|v\|_{\text{pow}}^2 := \limsup_{T \to \infty} \frac{1}{T} \int_0^T v^T(t) v(t) \, \mathrm{d}t$$

Notation

Power semi-norm

$$\|v\|_{\text{pow}}^2 \;:=\; \limsup_{T\to\infty} \frac{1}{T}\int_0^T v^T(t)v(t)\,\mathrm{d}t$$

For a given control $u(t)$ or $u(x)$ and disturbance $d(t)$

$$\begin{aligned} \dot{x} &= (A + K(u))\,x + B\,d \\ z_{ud} &:= \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix} \end{aligned}$$

Problem formulation

Design $u(t)$ to minimize worst-case induced power

$$\underset{u}{\text{minimize}} \quad \sup_{\|d\|_{\text{pow}}^2 \leq 1} \frac{\|z_{ud}\|_{\text{pow}}^2}{\|d\|_{\text{pow}}^2}$$

There is a $u(t) \equiv \bar{u}^*$ that achieves the minimum

Optimality of constant control input

Positive system → for $u(t) \equiv \bar{u}$, worst $d(t) \equiv \bar{d}$

$$\sup_{\|d\|_{\text{pow}}^2 \le 1} \frac{\|z_{\bar{u}d}\|_{\text{pow}}^2}{\|d\|_{\text{pow}}^2} = \|z_{\bar{u}\bar{d}}\|_{\text{pow}}^2$$

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If there is a $u(t)$ or $u(x)$ that outperforms $\bar{u}^*$,

$$\|z_{u\bar{d}}\|_{\text{pow}}^2 < \|z_{\bar{u}^*\bar{d}}\|_{\text{pow}}^2$$

This is not possible

Convexity of $\|z_{ud}\|_{\mathrm{pow}}^2$

$$\dot{x} \;=\; (A \,+\, K(u))\, x$$

Output at time t from initial condition x_0

$$c^T \mathrm{e}^{(A+K(u))t} x_0$$

convex in $u(t)$ (Colaneri et al '14)

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convex in $u(t)$ (Colaneri et al '14)

$$\|z_{u\bar{d}}\|^2_{\text{pow}} \;=\; \limsup_{T \to \infty} \frac{1}{T} \int_0^T x^T(t)\, Q\, x(t) \;+\; u^T(t)\, R\, u(t)\, \mathrm{d}t$$

convex in $u(t)$

Directional derivatives of $\|z_{u\bar{d}}\|_{\text{pow}}^2$

Perturb $\bar{u}$ with zero-mean periodic signal $\tilde{u}$

$$u(t) \;=\; \bar{u} \;+\; \varepsilon\tilde{u}(t), \qquad x(t) \;=\; \bar{x}(t) \;+\; \varepsilon\tilde{x}(t) \;+\; H.O.T.$$

Dynamics

$$0 \;=\; -\dot{x} \;+\; (A \;+\; K(\bar{u} \;+\; \varepsilon\tilde{u}))\,x \;+\; B\bar{d}$$

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Dynamics

$$\begin{aligned}
0 \;&=\; -\dot{x} \;+\; (A \;+\; K(\bar{u} \;+\; \varepsilon\tilde{u}))\,x \;+\; B\bar{d} \\
0 \;&=\; \Big[-\;\dot{\bar{x}} \;+\; (A \;+\; K(\bar{u}))\bar{x} \;+\; B\bar{d}\Big] \\
&\quad +\varepsilon\;\Big[-\;\dot{\tilde{x}} \;+\; (A \;+\; K(\bar{u}))\tilde{x} \;+\; K(\tilde{u})\bar{x}\Big] \\
&\quad +\;\; H.O.T.
\end{aligned}$$

Directional derivatives of $\|z_{u\bar{d}}\|^2_{\mathrm{pow}}$

$$\dot{\bar{x}} = (A + K(\bar{u}))\bar{x} + B\bar{d}$$

$\bar{x}(t)$ asymptotically constant

Directional derivatives of $\|z_{u\bar{d}}\|^2_{\text{pow}}$

$$\dot{\bar{x}} = (A + K(\bar{u}))\bar{x} + B\bar{d}$$

$\bar{x}(t)$ asymptotically constant

$$\dot{\tilde{x}} = (A + K(\bar{u}))\tilde{x} + K(\tilde{u})\bar{x}$$

Linear system with zero-mean periodic forcing $\rightarrow$ $\tilde{x}$ zero-mean periodic

Directional derivatives of $\|z_{ud}\|^2_{\text{pow}}$

$$\dot{\bar{x}} = (A + K(\bar{u}))\bar{x} + B\bar{d}$$

$\bar{x}(t)$ asymptotically constant

$$\dot{\tilde{x}} = (A + K(\bar{u}))\tilde{x} + K(\tilde{u})\bar{x}$$

Linear system with zero-mean periodic forcing $\rightarrow \tilde{x}$ zero-mean periodic

$$\|z_{u\bar{d}}\|^2_{\text{pow}} = \|z_{\bar{u}\bar{d}}\|^2_{\text{pow}} + \underbrace{2\varepsilon \left\langle R^{1/2}\bar{u}, R^{1/2}\tilde{u} \right\rangle + 2\varepsilon \left\langle Q^{1/2}\bar{x}, Q^{1/2}\tilde{x} \right\rangle}_{=0} + H.O.T.$$

Therefore, $\partial_{\tilde{u}} \|z_{\bar{u}\bar{d}}\|^2_{\text{pow}} = 0 \rightarrow \bar{u}^*$ global minimum

Optimality of $\bar{u}^*$

Besicovitch almost periodic functions ($\mathcal{B}^2$): closure of span of

$$\alpha\, \mathrm{e}^{j\lambda t}, \qquad \alpha \in \mathbb{C}^n, \lambda \in \mathbb{R}$$

with respect to $\|f - g\|_{\mathrm{pow}}$

Constant $\bar{u}^*$ optimal over

- All $u(t) \in \mathcal{B}^2$
- All $u(x)$ for which $u(x(t)) \in \mathcal{B}^2$

Optimal control problem

Convex problem

$$\underset{\bar{u}}{\text{minimize}} \;\; \bar{\sigma}^2 \left(Q^{\frac{1}{2}} \left(A \; + \; D(\bar{u})\right)^{-1} B \right) \; + \; \bar{u}^T R \bar{u}$$

- Subgradient descent
- continuously differentiable if graph of A strongly connected

(Dhingra, Colombino and Jovanović, ECC 16)

Model and input uncertainty

Drug treatment models have large uncertainty

$$\dot{x} = ((A + \Delta_A) + K(\bar{u} + \delta_u))\,x + Bd$$

Unknown virus mutation dynamics

$$\Delta_A = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1n} \\ \vdots & \ddots & \vdots \\ \delta_{n1} & \cdots & \delta_{nn} \end{bmatrix}, \qquad |\delta_{ij}| \leq \alpha_{ij}$$

Unknown drug delivery

$$|(\delta_u)_k| \leq \beta_k$$

Robust control problem

Assume highest mutation probabilities and lowest drug efficacy

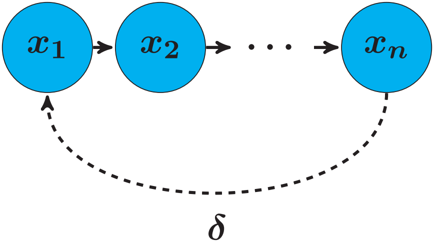
$$\bar{A} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$$

Convex problem

$$\underset{\bar{u}}{\text{minimize}} \; \bar{\sigma}^2 \left(Q^{\frac{1}{2}} \left(A + |\bar{A}| + K(\bar{u} - \beta) \right)^{-1} B \right) + \bar{u}^T R \bar{u}$$

(Colombino and Smith, TAC ‘16))

Robustness example



$$\underbrace{\begin{bmatrix} 1 & & & \delta \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}}_{A}$$

$$K(u) = -u\,I, \qquad Q = I,\; R = 3I, \qquad \delta \leq 0.1$$

Stability margin for δ: $(1 - u)^n$

$n = 10$	Nominal	Robust
$\bar{u}^*$	1.5936	1.9413
margin	0.0054	0.5461

Ongoing work

- Characterize feedback laws that yield $u(x(t)) \in \mathcal{B}^2$
- Extension of $\mathcal{H}_2$ performance metric
- Time-varying positive systems $A(t)$

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