Sparsity-promoting optimal control of systems with invariances and symmetries

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Standard feedback control

$$
\begin{aligned}
\dot{x} &= A\,x + B\,u + d \\
z &= \begin{bmatrix} I \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u
\end{aligned}
$$

Design state feedback gain matrix $u = -Fx$

$$
\dot{x} = (A - B\,F)\,x + d
$$

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controllers known when F unconstrained

Problem formulation

$$\dot{x} = \big(A - K(v)\big)\, x + d$$

- Control parameter $v \in \mathbb{R}^m$
- $K \in \mathbb{R}^{n \times n}$ linear function of v

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Objectives:

- Optimize closed-loop performance
- Impose structure on v

Related work

Design of system dynamics

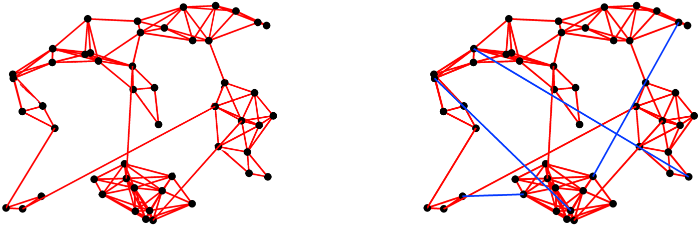
- Convex monotone systems (Rantzer and Bernhardsson '14)
- Combination drug therapy (Jonsson, Rantzer, Matni, and Murray '14)
- Vibrational control of bridges (Nelson, Rajamani, Gastineau, Schultz, and Wojtkiewicz '15)
- $\mathcal{H}_2$ control of symmetric systems (Dhingra and Jovanović '15)
- Decentralized control of positive systems (Dhingra, Colombino, and Jovanović '16)

Sparse feedback synthesis

- Via nonlinear programming (Lin, Fardad, and Jovanovic '13)
- Decentralized control (Schuler, Münz, Allgöwer '14)
- Positive systems (Tanaka and Langbort '11)
- Spatially-invariant state feedback (Zoltowski, Dhingra, and Jovanović '14)
- Quadratically invariant systems (Matni '15)

Examples – Growing consensus networks

$$\dot{x} \;=\; -\big(L \;+\; E\,\mathrm{diag}\{v\}\,E^T\big)\;x \;+\; d$$

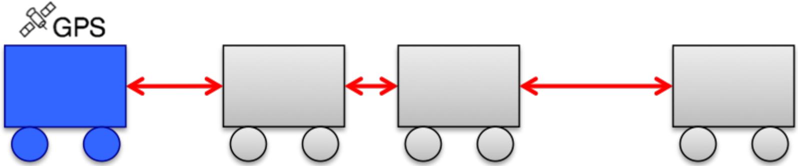


- Nodes x_i determine average via relative information exchange
- L is graph laplacian, E specifies edges, v specifies edge weights

Lin, Fardad, and Jovanovic, Allerton '12
Hassan-Moghaddam and Jovanovic, TCNS '16 (submitted);
also arXiv:1506.03437v2

Examples – Leader selection

$$\dot{x} = -(L + \mathrm{diag}\{v\})\, x + d$$

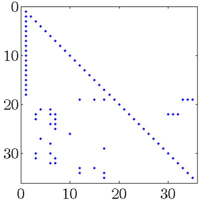


- Some nodes are 'leaders' with access to absolute measurements
- L is graph laplacian, v specifies leaders

Fitch and Leonard, CDC '13
Lin, Fardad, and Jovanovic, TAC '14

Examples – Combination drug therapy

$$\dot{x} = \left(A - \sum_{k=1}^{m} v_k D_k \right) x + d$$



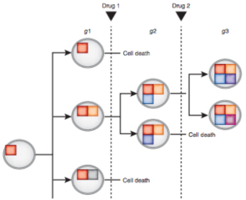


Image credit: Al-Lazikani et al '12

- Mutagen x_i mutates to x_j at rate A_{ji}
- Drug v_k kills x_i at rate $(D_k)_{ii}$

Rantzer and Bernhardsson CDC '14

Jonsson, Rantzer, Matni, and Murray CDC '14

Regularization

minimize $\quad J(v) \quad + \quad \gamma \; g(v)$

$\downarrow$ $\mathcal{H}_2$ **or** $\mathcal{H}_\infty$ **norm**

$\downarrow$ **structure-promoting regularizer**

Regularization

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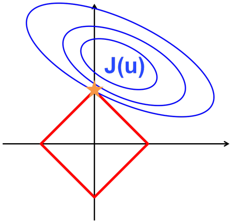
$\downarrow$ **$\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm**

$\downarrow$ **structure-promoting regularizer**



- e.g. $g(v) = \sum |v_i|$ for sparsity
- γ specifies importance of sparsity
- $\gamma = 0$ yields dense v

Fardad, Lin, and Jovanovic ACC '11

Lin, Fardad, Jovanovic TAC '13

$\mathcal{H}_2$ norm of system

$$\dot{x} \;=\; \big(A \;-\; K(v)\big)\, x \;+\; d$$

$\mathcal{H}_2$ norm

$$J_2(v) \;:=\; \text{trace}\,(X(v))$$

$X(v)$ is state covariance

$$\big(A \;-\; K(v)\big)\, X \;+\; X\,\big(A \;-\; K(v)\big)^T \;+\; I \;=\; 0$$

In general, $J_2(v)$ nonconvex

$\mathcal{H}_2$ norm of symmetric system

$$\dot{x} \;=\; \big(A_s \;-\; K_s(v)\big)\, x \;+\; d$$

A_s and $K_s(v)$ symmetric $\Longrightarrow$ explicit solution

$$X_s \;=\; -\,\frac{1}{2}\,\big(A_s \;-\; K_s(v)\big)^{-1}$$

$$\boxed{J_{2s}(v) \text{ convex}}$$

$\mathcal{H}_2$ norm of symmetric system

$$\dot{x} = \big(A_s - K_s(v)\big)\, x + d$$

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$$\boxed{J_{2s}(v) \text{ convex}}$$

SDP characterization

$$\begin{array}{ll} \text{minimize} & \text{trace}\,(X_s) + g(v) \\ \text{subject to} & \begin{bmatrix} X_s & I \\ I & 2(K_s(v) - A_s) \end{bmatrix} \succ 0 \end{array}$$

$\mathcal{H}_\infty$ norm of system

$$\dot{x} \;=\; \big(A \,-\, K(v)\big)\, x \,+\, d$$

$\mathcal{H}_\infty$ norm

$$J_\infty(v) \;:=\; \sup_{\|d\|_{\mathcal{L}_2} \le 1} \frac{\|x\|_{\mathcal{L}_2}}{\|d\|_{\mathcal{L}_2}}$$

Peak of frequency response

$$J_\infty(v) \;=\; \sup_{\omega}\; \sigma_{\max}\big((j\omega I \,-\, (A \,-\, K(v)))^{-1}\big)$$

In general, $J_\infty(v)$ nonconvex and nondifferentiable

$\mathcal{H}_\infty$ norm of symmetric system

$$\dot{x} \;=\; \big(A_s \;-\; K_s(v)\big)\,x \;+\; d$$

A_s and $K_s(v)$ symmetric $\implies$ peak at $\omega = 0$

$$J_{\infty s}(v) \;=\; \sigma_{\max}\left(-(A_s \;-\; K_s(v))^{-1}\right)$$

$$\boxed{J_{\infty s}(v) \text{ convex}}$$

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Exploiting symmetry

Central Idea:

Use symmetric components

$$A_s := \frac{1}{2}(A + A^T), \qquad K_s(v) := \frac{1}{2}(K(v) + K^T(v))$$

to inform design for original system

- Convex characterization
- Relationship between A_s and A
  - Stability
  - Performance
  - Fidelity

Stability

$$A_s \text{ Hurwitz} \implies A \text{ Hurwitz}$$

Not a necessary condition

Implication

Designing v for A_s guarantees stability

$$\underbrace{\dot{x} \; = \; \big(A_s \; - \; K_s(v)\big)\, x \; + \; d}_{\text{stable}} \quad \implies \quad \underbrace{\dot{x} \; = \; \big(A \; - \; K(v)\big)\, x \; + \; d}_{\text{stable}}$$

Upper bound on performance

Symmetric system's $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms upper bound original

$$\underbrace{\dot{x} \;=\; A\,x \;+\; d}_{\begin{array}{c}\|\cdot\|_2\\ \|\cdot\|_\infty\end{array}} \quad \begin{array}{c}\leq\\ \leq\end{array} \quad \underbrace{\dot{x} \;=\; A_s\,x \;+\; d}_{\begin{array}{c}\|\cdot\|_2\\ \|\cdot\|_\infty\end{array}}$$

Convex upper bound on performance

Implication

A controller v designed for A_s will perform better with A

$$\underbrace{\dot{x} \;=\; \big(A \;-\; K(v)\big)\,x \;+\; d}_{\begin{array}{c}J_2(v)\\ J_\infty(v)\end{array}} \quad \begin{array}{c}\leq\\ \leq\end{array} \quad \underbrace{\dot{x} \;=\; \big(A_s \;-\; K_s(v)\big)\,x \;+\; d}_{\begin{array}{c}J_{2s}(v)\\ J_{\infty s}(v)\end{array}}$$

Small perturbations

- Perturb system dynamics of

$$\dot{x} = A_s x + d$$

by antisymmetric matrix A_a

$$A = A_s + \epsilon A_a$$

- First order correction to both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms is 0

Implication

A_s is a very good approximation when A is close to symmetric

$$\begin{aligned} J_2(v) &= J_{2s}(v) + O(\epsilon^2) \\ J_\infty(v) &= J_{\infty s}(v) + O(\epsilon^2) \end{aligned}$$

Design procedure

- Form $A_s = \frac{1}{2}(A + A^T)$ and $K_s(v) = \frac{1}{2}(K(v) + K(v)^T)$

- Solve convex symmetric design problem

$$
\begin{aligned}
v_2^* &= \text{argmin} \quad J_{2s}(v) \; + \; \gamma \, g(v) \\
v_\infty^* &= \text{argmin} \quad J_{\infty s}(v) \; + \; \gamma \, g(v)
\end{aligned}
$$

- Implement v^* on original system A

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- Implement v^* on original system A

- Note: can jointly minimize/constrain $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance

Undirected edge addition in directed consensus networks

$$\dot{x} \;=\; -\left(L \;+\; E\,\mathrm{diag}\{v\}\,E^T\right)\,x \;+\; d$$

Regularized problem

$$\text{minimize} \quad J_s(v) \;+\; v^T v \;+\; \gamma \sum |v_i|$$

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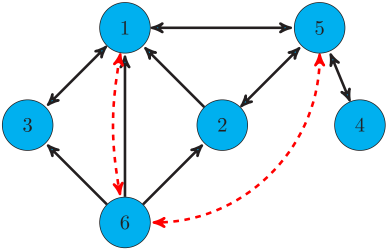
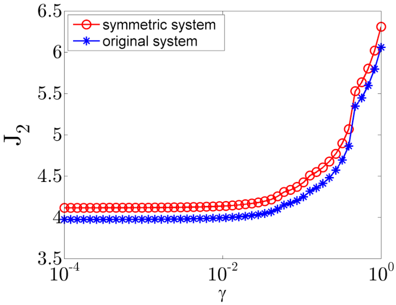
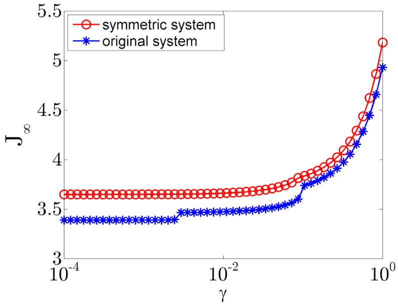


Edges added for $\gamma = 1$

Undirected edge addition in directed consensus networks





Closed-loop performance with respect to γ

Computational advantage from invariances

SDPs scale poorly with constraint size

$$\begin{bmatrix} X & I \\ I & -A \end{bmatrix} \succeq 0$$

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Jointly block-diagonalize A and K

$$PAP^T = \text{blkdiag}\{\hat{A}_i\}, \qquad PK(v)P^T = \text{blkdiag}\{\hat{K}_i(v)\}$$

E.g. spatially-invariant control of spatially-invariant PDEs

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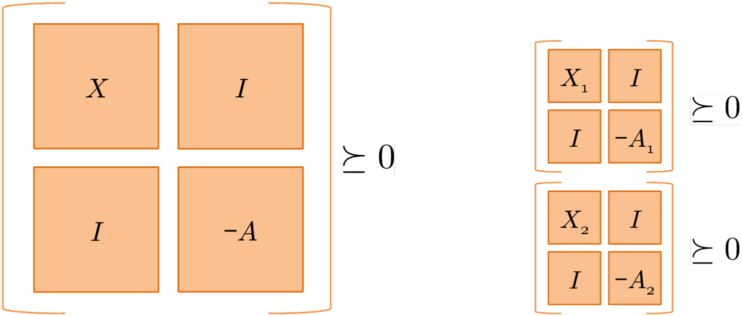
SDPs scale poorly with constraint size



Jointly block-diagonalize A and K $\implies$ many small SDP constraints

$$PAP^T = \text{blkdiag}\{\hat{A}_i\}, \qquad PK(v)P^T = \text{blkdiag}\{\hat{K}_i(v)\}$$

E.g. spatially-invariant control of spatially-invariant PDEs

Swift-Hohenberg equation

$$\partial_t \psi(t,x) \;=\; \beta\, \psi(t,x) \,-\, (1+\partial_{xx})^2\, \psi(t,x) \,-\, v(x)\psi(t,x)$$

Swift-Hohenberg equation

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Discretized at N points $\implies \psi \in \mathbb{R}^N$

$$\dot{\psi} \;=\; (\beta I \,-\, (I+D^2)^2 \,-\, V)\psi$$

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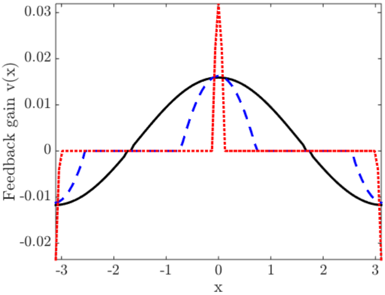
Spatial DFT $\implies N$ decoupled $\psi_\kappa \in \mathbb{R}$

$$\dot{\hat{\psi}}_\kappa \;=\; (\beta-(1-\kappa^2)^2 \,-\, \hat{v}_\kappa)\hat{\psi}_\kappa$$

Swift-Hohenberg equation

$$\text{minimize} \quad J_2(V) \; + \; \|V\|_F^2 \; + \; \gamma \sum |V_{ij}|$$



Left: $v(x)$ with $N = 101$ for $\gamma = \mathbf{0}$, $\gamma = \mathbf{0.1}$ and $\gamma = \mathbf{10}$

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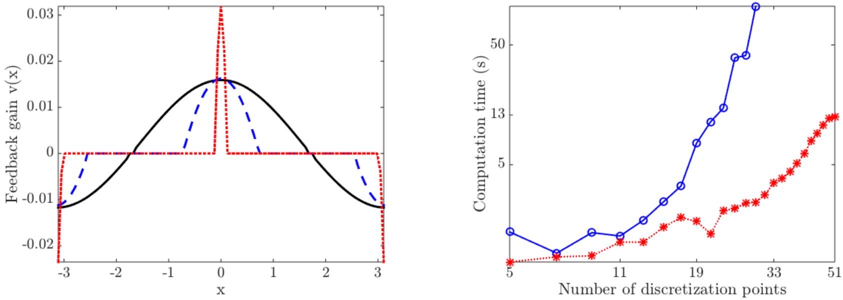

Left: $v(x)$ with $N = 101$ for $\gamma = \mathbf{0}$, $\gamma = \mathbf{0.1}$ and $\gamma = \mathbf{10}$

Right: Computation time in V and $\hat{v}$ coordinates

Conclusions

Regularized design

- Exploit convex characterization using symmetric component
- Implement on original system
  - Stability guarantee
  - Upper bound on performance
  - Good approximation when 'mostly' symmetric

Exploiting invariance

- A and K jointly block-diagonalizable
- Linear scaling with number of SDP blocks

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