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Sparsity-promoting optimal control of systems with invariances and symmetries

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Abstract: We take advantage of system invariances and symmetries to gain convexity and computational advantage in regularized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems. For systems with symmetric dynamic matrices, the problem of minimizing the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance of the closed-loop system can be cast as a convex optimization problem. Although the assumption of symmetry is restrictive, studying the symmetric component of a general system's dynamic matrices provides bounds on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of the original system. Furthermore, we show that for certain classes of systems, block-diagonalization of the system matrices can bring the regularized optimal control problems into forms amenable to efficient computation via distributed algorithms. One such class of systems is spatially-invariant systems, whose dynamic matrices are circulant and therefore block-diagonalizable by the discrete Fourier transform.

Keywords: Convex synthesis, $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control, sparse controller, sparsity-promoting optimal control, spatially-invariant systems, structured design, symmetry.

1. INTRODUCTION

Structured control problems are, in general, challenging and nonconvex. Many recent works have identified classes of systems for which structured optimal control problems can be cast in convex forms. These include funnel causal and quadratically invariant systems (Bamieh and Voulgaris, 2005; Rotkowitz and Lall, 2006), positive systems (Rantzer, 2011; Dhingra et al., 2016), structured and sparse consensus/synchronization networks (Xiao et al., 2007; Lin et al., 2012; Fardad et al., 2014a; Wu and Jovanović, 2014; Hassan-Moghaddam and Jovanović, 2015), optimal sensor/actuator selection (Polyak et al., 2013; Dhingra et al., 2014), and symmetric modifications to symmetric linear systems (Dhingra and Jovanović, 2015).

In many large-scale problems, controller structure is vitally important. As such, much effort has been devoted to developing scalable algorithms for nonconvex regularized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design problems (Fardad et al., 2011; Lin et al., 2013; Schuler et al., 2011; Polyak et al., 2013; Dhingra et al., 2014; Matni and Chandrasekaran, 2014; Matni, 2015; Matni and Chandrasekaran, 2015). Although many recent works have developed efficient algorithms for the nonconvex regularized $\mathcal{H}_2$ problems, in general, regularized $\mathcal{H}_\infty$ problems are difficult because the $\mathcal{H}_\infty$ norm is nonsmooth.

We propose a principled approach to general regularized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design. Our formulation treats control problems that minimize the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm by modifying the dynamical generator of a linear system, such as in linear state feedback. In this work, we use symmetries in system structure to form convex problems and gain computational advantage.

The contributions of this paper are twofold. First, in a similar vein as (Dhingra and Jovanović, 2015), we utilize the symmetric component of a general linear system to form a symmetric system for which the regularized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems are convex. We implement the controllers designed by this method on the original system. We show that this procedure guarantees stability and that the closed-loop $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of the symmetric system is an upper bound on the closed-loop $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of the original system.

Second, we provide a way to gain computational advantage by exploiting the block-diagonalizability of large scale systems. Such a structure arises, for example, in spatially-invariant systems (Bamieh et al., 2002). In (Zoltowski et al., 2014), the authors took advantage of this property to develop an efficient and scalable algorithm for sparsity-promoting feedback design. When a spatially-invariant system is subject to a spatially-invariant control law, the dynamics of the system can be represented as the sum of independent subsystems, making the problem amenable to distributed optimization.

The rest of the paper is organized as follows. In Section 2, we formulate the regularized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems and provide several example applications that motivate our developments. In Section 3, we describe the symmetric design procedure and provide several results on stability, spectral properties, performance guarantees, and first order approximations. In Section 4, we develop methods which gain computational advantage from jointly

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block-diagonalizable systems and describe this process for spatially-invariant systems in particular. In Section 5, we provide two examples. Finally, in Section 6, we summarize our work and discuss ongoing research directions.

2. PROBLEM FORMULATION

We consider the class of systems,

$$\begin{aligned} \dot{x} &= (A - K(v))\,x + Bd \\ z &= \begin{bmatrix} C \\ R(v) \end{bmatrix} x \end{aligned} \tag{1}$$

where $v \in \mathbb{R}^m$ is a design parameter, $K(\cdot) : \mathbb{R}^m \to \mathbb{R}^{n \times n}$ is a linear operator, $x(t) \in \mathbb{R}^n$ is the state vector, C is mapping from the state to a regulated output, $R(v)$ is a mapping from the state to a measure of control effort, $d(t) \in \mathbb{R}^p$ is a white stochastic disturbance with $\mathbf{E}\,(d(t_1)\,d^T(t_2)) = I\delta(t_1 - t_2)$, and $\mathbf{E}$ is the expectation operator. Taking $v = \mathrm{vec}(F)$, $K(v) = B_2 F$ and $R(v) = R^{1/2} F$ where $R \succeq 0 \in \mathbb{R}^{p \times p}$, $F \in \mathbb{R}^{p \times n}$, and $B_2 \in \mathbb{R}^{n \times p}$ yields the traditional state feedback control problem. We consider v to be constant in time.

Our objective is to design a stabilizing v that solves the regularized optimal control problem,

$$\begin{array}{ll} \underset{v}{\text{minimize}} & J(v) + g(v) \\ \text{subject to} & A - K(v) \;\; \text{Hurwitz} \end{array} \tag{2}$$

where $J(v)$ is a performance metric, taken to be either the closed loop $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm, and $g(v)$ can be any convex function of v. The $\mathcal{H}_2$ performance, which we denote by $J_2(v)$, is a measure of the variance amplification from the disturbances d to the regulated output z in system (1),

$$J_2(v) := \lim_{t \to \infty} \mathbf{E}\left(z^T(t)\,z(t)\right)$$

which can be computed by

$$J_2(v) = \mathrm{trace}\left(X(C^T C + R^T(v) R(v))\right)$$

where X is the controllability gramian

$$(A - K(v))\,X + X\,(A - K(v))^T + BB^T = 0.$$

The $\mathcal{H}_\infty$ performance metric, which we denote by $J_\infty(v)$, is the maximum induced $\mathcal{L}_2$ gain from d to z in system (1),

$$J_\infty(v) := \sup_{\|d\|_{\mathcal{L}_2} \le 1} \frac{\|z\|_{\mathcal{L}_2}}{\|d\|_{\mathcal{L}_2}},$$

where the $\mathcal{L}_2$ norm of a signal f is defined as,

$$\|f\|_{\mathcal{L}_2}^2 := \int_0^\infty f^2(t)\,\mathrm{d}t.$$

This performance metric corresponds to the peak of the frequency response,

$$J_\infty(v) = \sup_{\omega} \sigma_{\max}\left(C\,(j\omega I - (A + K(v)))^{-1}\,B\right).$$

The unregularized $\mathcal{H}_2$ and $\mathcal{H}_\infty$-optimal linear state feedback problems can be cast in a convex form via a suitable change of coordinates; however, this change of coordinates does not preserve the structure of the design variable v.

For many applications, v has physical significance and penalizing it directly via $g(v)$ is desirable. For example, a quadratic penalty, $\|v\|_2^2$, would limit the magnitude of v, and an ℓ_1 penalty, $\|v\|_1 := \sum_i |v_i|$, would promote sparsity.

Many structured optimal control problems can be cast in the form of (2). For example, structured state feedback problems have been extensively studied with particular applications to consensus networks and power systems (Lin et al., 2013; Wu and Jovanović, 2014; Wu et al., 2014, 2016, 2015; Dhingra and Jovanović, 2016). Two other applications are given below.

2.1 Applications

Design of edges in networks The problem of adding undirected edges to an existing network can be cast in this problem form. The dynamics are,

$$\dot{x} = -\left(L + E\,\mathrm{diag}(v) E^T\right)\,x + d$$

where L is a directed graph Laplacian which contains information about how the nodes are connected, E contains information about the locations of potential added edges, and $K(v) := E\,\mathrm{diag}(v) E^T$ is a diagonal matrix of added edge weights (Fardad et al., 2014b). Taking the regularizer to be the ℓ_1 norm $g(v) = \sum_i |x_i|$ would limit the number of edges added to the network.

Combination drug therapy design for HIV treatment The problem of designing drug dosages for treating HIV (Jonsson et al., 2014, 2013) can be cast as,

$$\dot{x} = \left(A - \sum_{k=1}^{m} v_k D_k\right)\,x + d.$$

Here, the elements of x represent populations of HIV mutants. The diagonal elements of A represent each mutant's replication rate and the off diagonal elements of A represent the probability of mutation from one mutant to another. The components of the vector v are dosages of different drugs, where D_k is a diagonal matrix containing information about how efficiently drug k kills each HIV mutant. Here, quadratic regularization $g(v) = \|v\|_2^2$ would limit the dose of the drugs prescribed and ℓ_1 regularization $g(v) = \sum_i |x_i|$ would limit the amount of drugs prescribed.

3. SYMMETRIC SYSTEM DESIGN

One class of system for which $J_2(v)$ and $J_\infty(v)$ are convex arises when $B = C = I$, and A and $K(v)$ are symmetric matrices. Although this assumption seems restrictive, studying such systems can inform the design of structured controllers for more general classes of systems.

Any matrix A can be decomposed into its symmetric $A_s = \frac{1}{2}(A + A^T)$ and antisymmetric $A_a = \frac{1}{2}(A - A^T)$ components. The system which corresponds to the symmetric components of the general system (1),

$$\dot{x} = (A_s - K_s(v))\,x + d \tag{3}$$

where $K_s(v) = \frac{1}{2}(K(v) + K^T(v))$ reveals interesting characteristics of the original system.

In this section, we first show the convex formulations that correspond to the optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design of symmetric systems. We then establish stability guarantees and performance bounds for applying controllers designed by solving the convex problem of regularized optimal control on the symmetric system (3) to the original system (1). Finally, we use perturbation analysis to show that the symmetric system is a high fidelity approximation for systems which are dominated by the symmetric component.

3.1 Convex optimal control for symmetric systems

Although more general symmetric systems can be cast as convex problems, here we assume $B = C = I$ and $R(v) = 0$ to facilitate the transition to the discussion of spectral properties and performance bounds.

$\mathcal{H}_2$-optimal control When $A_s = A_s^T$ is symmetric, the controllability gramian of system (3) can be explicitly expressed as,

$$X_s \;=\; -\,\frac{1}{2}\,(A_s \;-\; K_s(v))^{-1}$$

and, by taking a Schur complement, the regularized optimal $\mathcal{H}_2$ control problem can be cast in a convex function of v and an auxiliary variable Θ,

$$\begin{array}{ll} \underset{v,\Theta}{\text{minimize}} & \frac{1}{2}\text{trace}(\Theta) \;+\; g(v) \\ \text{subject to} & \begin{bmatrix} \Theta & I \\ I & -A_s + K_s(v) \end{bmatrix} \succ 0. \end{array} \tag{4}$$

The matrix $A_s - K_s(v)$ is always invariable when it is Hurwitz. We note the structured LQR problem (i.e., $R(v) = R^{1/2}K_s(v)$) for symmetric systems can also be expressed as an SDP by taking the Schur complement of $K_s(v)RK_s(v)$.

$\mathcal{H}_\infty$-optimal control The peak of the frequency response of a symmetric system occurs at $\omega = 0$.

Proposition 1. For a system (3) with symmetric dynamics, the disturbance that achieves the maximum induced $\mathcal{L}_2$ amplification corresponds to the constant signal $d(t) = v$ where v is the right principal singular vector of A^{-1}.

Proof. A symmetric matrix can be diagonalized as, $A_s = U\Lambda U^T$ where Λ is a diagonal matrix with the eigenvalues of A_s on the main diagonal and the columns of U contain the corresponding eigenvectors. For such a matrix,

$$(j\omega I \;-\; A_s)^{-1} \;=\; U\,\text{diag}\left\{\frac{1}{j\omega \;-\; \lambda_i}\right\}U^T.$$

It is clear that $\omega = 0$ maximizes the singular values of the above matrix. Thus, the $\mathcal{H}_\infty$ norm of (3) can be characterized by $\sigma_{\max}\left(-(A_s \;-\; K(v))^{-1}\right)$.

The $\mathcal{H}_\infty$-optimal control problem for symmetric systems can therefore be expressed as,

$$\begin{array}{ll} \underset{v,\Theta}{\text{minimize}} & \sigma_{\max}(\Theta) \;+\; g(v) \\ \text{subject to} & \begin{bmatrix} \Theta & I \\ I & -A_s + K_s(v) \end{bmatrix} \succeq 0. \end{array} \tag{5}$$

As we show in the next subsection, this convex problem can be used for structured $\mathcal{H}_\infty$ control design. This is particularly advantageous because many of the existing algorithms for general structured $\mathcal{H}_2$ control cannot be extended to the structured $\mathcal{H}_\infty$ problem.

3.2 Stability and performance guarantees

Stability of the symmetric system (3) implies stability of the corresponding original system (1).

Lemma 2. (Dhingra and Jovanović (2015, Lemma 1)). Let the symmetric part of A, $A_s := (A + A^T)/2$, be Hurwitz. Then, A is Hurwitz.

Remark 1. This is not a necessary condition; A may be Hurwitz even if A_s is not.

Performance guarantees The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the symmetric system are upper bounds on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the original system.

Proposition 3. [Dhingra and Jovanović (2015, Cor. 3)] When the systems (1) and (3) are stable, the $\mathcal{H}_2$ norm of the general system (1) is bounded from above by the $\mathcal{H}_2$ norm of the symmetric system (3).

We show that an analagous bound holds for the $\mathcal{H}_\infty$.

Proposition 4. When the systems (1) and (3) are stable, the $\mathcal{H}_\infty$ norm of the general system (1) is bounded from above by the $\mathcal{H}_\infty$ norm of the symmetric system (3).

Proof. From the bounded real lemma (Dullerud and Paganini, 2013), the $\mathcal{H}_\infty$ norm of the general system (1) is less than γ if there exists a $P \succ 0$ such that,

$$A^TP \;+\; PA \;+\; I \;+\; \gamma^{-2}P^2 \;\prec\; 0.$$

From Proposition 1, for the symmetric system (3), $\gamma > \sigma_{\max}(A_s^{-1})$. Taking $P = \gamma I$ for any $\gamma > \sigma_{\max}(A_s^{-1})$ and substituting it into the above linear matrix inequality (LMI) applied to the symmetric system (3) yields,

$$2\gamma A_s \;+\; 2I \;\prec\; 0.$$

Since A_s is Hurwitz, $A_s \prec 0$. Since $\gamma > -\lambda_{\max}(A_s^{-1})$, $\gamma^{-1} < -\lambda_{\min}(A_s)$, so $A_s \prec -\gamma^{-1}I$. Therefore the LMI is satisfied. Since $A_a = -A_a^T$, setting $P = \gamma I$ implies,

$$A^TP \;+\; PA \;=\; 2\gamma A_s$$

therefore substituting P into the bounded real lemma LMI for the general system (1), where $A = A_s + A_a$, yields,

$$A^TP \;+\; PA \;+\; I \;+\; \gamma^{-2}P^2 \;=\; 2\gamma A_s \;+\; 2I \;\prec\; 0.$$

3.3 Approximation bounds

In addition to being an upper bound, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the symmetric (3) and full (1) systems are close when A is dominated by the symmetric component.

Proposition 5. [Dhingra and Jovanović (2015, Prop. 4)] Let A_n be a normal matrix. The $O(\epsilon)$ correction to the $\mathcal{H}_2$ norm of the system

$$\dot{x} \;=\; A_n x \;+\; d$$

from an $O(\epsilon)$ antisymmetric perturbation A_a is zero.

We show that a similar property holds for the $\mathcal{H}_\infty$ norm.

Proposition 6. Let A_s be a symmetric matrix. The $O(\epsilon)$ correction to the $\mathcal{H}_\infty$ norm of the system

$$\dot{x} \;=\; A_s x \;+\; d$$

from an $O(\epsilon)$ antisymmetric perturbation A_a is zero.

Proof. From Proposition 1, the $\mathcal{H}_\infty$ norm of the symmetric system is given by $\sigma_{\max}(-A_s^{-1})$. The maximum singular value of a matrix is equivalent to,

$$\sigma_{\max}(X) \;=\; \sup_{\|v\|_2 \le 1, \|w\|_2 \le 1} v^TXw.$$

Since A_s is symmetric, $w = v$. Taking an $O(\epsilon)$ antisymmetric perturbation A_a to the above expression,

$\sigma_{\max}(-(A_s + \epsilon A_a)^{-1}) \approx -v^T A_s^{-1} v + \epsilon v^T A_s^{-1} A_a A_s^{-1} v$. Since A_a is antisymmetric, $\langle A_s^{-1} v v^T A_s^{-1}, A_a \rangle = 0$.

4. COMPUTATIONAL ADVANTAGES FOR STRUCTURED PROBLEMS

Structured control is often of interest for large-scale systems. As such, the computational scaling of algorithms used to compute optimal controllers is very important. In this section, we identify a class of systems which are amenable to scalable distributed algorithms.

When A and $K(v)$ are always simultaneously block-diagonalizable, the dynamics of the system can be expressed as the sum of independent subsystems. Define $\hat{x} := Px$ and let P be a unitary matrix such that,

$$\dot{\hat{x}} = (\hat{A} + \hat{K}(v))\hat{x}$$

where

$$\hat{A} := PAP^T, \quad \hat{K}(v) := PK(v)P^T,$$

and, for any choice of v, $\hat{A} + \hat{K}(v) = \text{blkdiag}\{\hat{A}_{11} + \hat{K}_{11}, \cdots, \hat{A}_{NN} + \hat{K}_{NN}\}$ is block-diagonal with N blocks of size $n \times n$ each.

For problems of this form, computing optimal control strategies is much more efficient in the $\hat{x}$ coordinates because the majority of the computational burden in solving problems (4) and (5) comes from the $nN \times nN$ LMI constraint involved in minimizing the performance metrics $J_2(v)$ or $J_\infty(v)$.

For this class of system, the $\mathcal{H}_2$-optimal control problem (4) can be expressed as,

$$\begin{aligned} &\underset{v,\Theta_i}{\text{minimize}} \ \frac{1}{2}\sum_i \text{trace}(\Theta_i) + g(v) \\ &\text{subject to} \begin{bmatrix} \Theta_i & I \\ I & -(\hat{A}_s)_{ii} + (\hat{K}_s(v))_{ii} \end{bmatrix} \succeq 0. \end{aligned} \tag{6}$$

which is an SDP with N separate $n \times n$ LMI blocks. Since SDPs scale with the sixth power of the LMI blocks, solving this reformulation scales with n^6 as opposed to $n^6 N^6$.

Analogously, the structured $\mathcal{H}_\infty$-optimal control problem (5) can be cast as,

$$\begin{aligned} &\underset{v,\Theta_i}{\text{minimize}} \ \max_i(\sigma_{\max}(\Theta_i)) + g(v) \\ &\text{subject to} \begin{bmatrix} \Theta_i & I \\ I & (\hat{A}_s)_{ii} - (\hat{K}_s(v))_{ii} \end{bmatrix} \succeq 0. \end{aligned} \tag{7}$$

One important class of system which satisfies these assumptions is spatially-invariant systems. This structure was used in (Zoltowski et al., 2014) to develop efficient techniques for sparse feedback synthesis.

4.1 Spatially-invariant systems

Spatially-invariant systems have a block-circulant structure which is block-diagonalizable by a Discrete Fourier Transform (DFT). A spatially-invariant system can be represented by N subsystems with n states each. The state vector $x \in \mathbb{R}^{nN}$ is composed of N subvectors $x_i \in \mathbb{R}^n$ which denotes the state of the subsystem. The matrix $A \in \mathbb{R}^{nN \times nN}$ is block-circulant with blocks of the size $n \times n$. For example, when $N = 3$,

$$A = \begin{bmatrix} A_0 & A_1 & A_{-1} \\ A_{-1} & A_0 & A_1 \\ A_1 & A_{-1} & A_0 \end{bmatrix}$$

where the blocks $\{A_0, A_{-1}, A_1\} \in \mathbb{R}^{n \times n}$.

It was shown in (Bamieh et al., 2002) that the optimal feedback controller for a spatially-invariant system is itself spatially-invariant. Assuming that the optimal *sparse* feedback controller is also spatially-invariant is equivalent to assuming that $K(v)$ is block-circulant. Block circulant matrices are block-diagonalizable by the appropriate DFT. Let the block Fourier matrix be

$$\Phi := \Phi_N \otimes I_n,$$

where I_n is the $n \times n$ identity matrix, Φ_N is the $N \times N$ discrete Fourier transform matrix, and $\otimes$ represents the Kronecker product. By introducing the change of variables $\hat{x} := \Phi x$, where

$$\hat{x} = \begin{bmatrix} \hat{x}_1^T & \cdots & \hat{x}_N^T \end{bmatrix}^T,$$

and $\hat{x}_i \in \mathbb{R}^n$, the original system's dynamics can be expressed as N independent $n \times n$ subsystems,

$$\hat{A} = \text{blkdiag}\{\hat{A}_{11}, \hat{A}_{22}, \hat{A}_{33}\}$$

Consequently, the optimal structured control problems (4) and (5) can be cast as the more formulations (6) and (7).

5. EXAMPLES

5.1 Directed Consensus Network

In this example, we illustrate the utility of the approach described in Section 3. Consider the network dynamics given by a directed network as described in Section 2.1.1,

$$\dot{x} = -(L + E\text{diag}(v)E^T)x$$

where L is a directed graph Laplacian, $K(v) = E\text{diag}(v)E^T$ represents the addition of undirected links, v is a vector that contains weights of these added links, and the incidence matrix E describes which edges may be added or altered. The regularization on v is given by,

$$g(v) = \|v\|_2^2 + \gamma \sum_i |v_i|$$

where the quadratic term limits the size of the edge weights, the ℓ_1 norm promotes sparsity of added links, and $\gamma > 0$ parametrizes the importance of sparsity.

For this concrete example, the network topology is given by Figure 1. The potential added edges can connect the following pairs of nodes: (1) − (2), (1) − (3), (1) − (5), (1) − (6), (2) − (5), (2) − (6), (3) − (6), and (4) − (5).

Controllers were designed by solving problems (4) and (5) for the symmetric version of the network over 50 log-distributed values of $\gamma \in [10^{-4}, 1]$. The closed-loop $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms obtained by applying these controllers to the symmetric and original systems are shown in Fig. 2. Fig. 1 also shows which edges were added for $\gamma = 1$.

5.2 Swift-Hohenberg Equation

Here we illustrate the utility of the block-diagonalization we describe in Section 4. Consider a particular realization

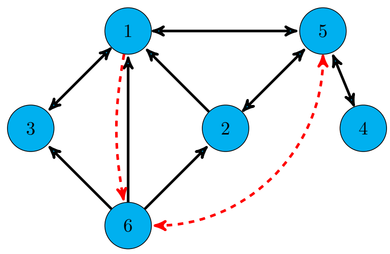

Fig. 1. Directed network (**black** solid arrows) with added undirected edges (**red** dashed arrows). Both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal structured control problems yielded the same set of added edges. In addition to these edges, the controllers tuned the weights of the edges $(1)-(3)$ and $(1)-(5)$.

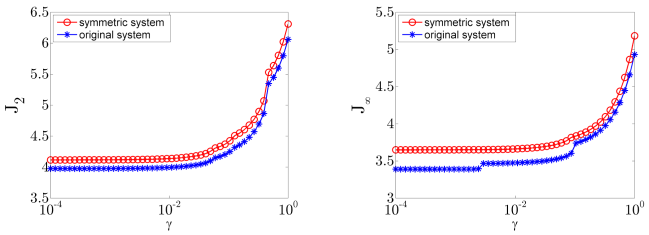

Fig. 2. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of the closed-loop symmetric system and the original system subject to a controller designed at various values of γ.

of the Swift-Hohenberg equation (Swift and Hohenberg, 1977),

$$\partial_t \psi(t,x) \;=\; \beta\,\psi(t,x) \;-\; (1+\partial_{xx})^2\;\psi(t,x) \;+\; v(x)\psi(t,x)$$

where $\beta \in \mathbb{R}$, and $\psi(t,\cdot), v(\cdot) \in L_2(-\infty,\infty)$, and $v(x)$ is a spatially-invariant feedback controller which is to be designed. A finite dimensional approximation of this system can be obtained by using the differentiation suite from (Weideman and Reddy, 2000) to discretize the problem into N points and approximating the infinite domain with periodic boundary conditions over the domain $L_2[0,2\pi]$. A sparse $\mathcal{H}_2$ feedback controller $v(x)$ can then be identified by solving problem (4).

We contrast this method with the approach we advocate in Section 4, where we use the DFT to decompose the system into N first-order systems corresponding to eigenfunctions of the Swift-Hohenberg equation and solve problem (6).

The state vector takes the form of $\psi(x)$ evaluated at grid points in x where the dynamics are given by,

$$\dot{\psi} \;=\; (A \;-\; V)\psi$$

where, $A = \beta I - (I + D^2)^2$. Here D is a discrete differentiation matrix from (Weideman and Reddy, 2000), and V is the circulant state feedback matrix.

Using the DFT over x, the Swift-Hohenberg equation can be expressed as a set of independent first-order systems,

$$\dot{\hat{\psi}}_x \;=\; (a_x \;-\; \hat{v}_x)\hat{\psi}_x$$

where $a_x := \beta - (1-\kappa_x^2)^2$, and the new coordinates are $\hat{\psi} := P\psi$, P is the DFT matrix, κ_x is the wavenumber (spatial frequency), and $\hat{v}$ represents V in the Fourier space; i.e., $V = P^T \text{diag}(\hat{v})P$.

We take the regularization term to be

$$g(v) \;=\; \|V\|_F^2 \;+\; \gamma\|V\|_1$$

where $\|X\|_1 := \sum_{ij}|X_{ij}|$ is the elementwise ℓ_1 norm and γ is a parameter which specifies the emphasis on sparsity relative to performance.

For the $\mathcal{H}_2$ problem, the regularized optimal control problem is of the form of (4) with $K_s(v) = V$ and V is circulant. In that formulation, the problem is an SDP with one $N \times N$ LMI block. In the Fourier space, the problem can be expressed as (6), which takes the particular form,

$$\begin{array}{ll} \underset{\hat{v}}{\text{minimize}} & \dfrac{1}{2}\sum \dfrac{1}{-a_x \,+\, \hat{v}_x} \;+\; g\left(P^T \text{diag}(\hat{v})P\right) \\ \text{subject to} & -\,a_x \,+\, \hat{v}_x \;\geq\; 0 \end{array}$$

which does not require the large SDP constraints in (4).

We solved the regularized $\mathcal{H}_2$ optimal control problem by solving the general formulation (4) and the more efficient formulation (6) for $\beta = 0.1$, $\gamma = 1$ and N varying from 5 to 51 using CVX, a general purpose convex optimization solver (Grant and Boyd, 2013).

Taking advantage of spatial invariance yields a significant computational advantage, as can be seen in Figure 3. Although both expressions of the problem yield the same solution, solving the realization in (6) is much faster and allows us to examine much larger problem dimensions. In Figure 4, we show the spatially-invariant feedback controller for one point in the domain, i.e., one row of V, computed for $N = 101$ at $\gamma = 0$, $\gamma = 0.1$, and $\gamma = 10.$.

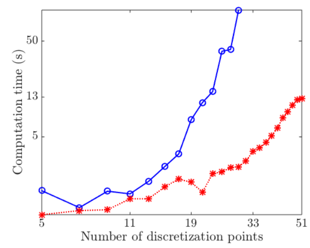

Fig. 3. Computation time for the general formulation (4) (**blue** $\circ$) and that which takes advantage of spatial invariance (6) (**red** $*$).

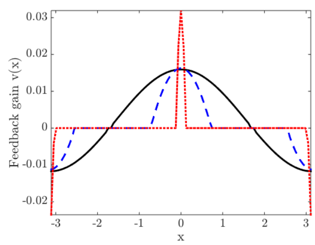

Fig. 4. Feedback gain $v(x)$ for the node at position $x = 0$, computed with $N = 51$ and $\gamma = 0$ (**black** solid), $\gamma = 0.1$ (**blue** dashed), and $\gamma = 10$ (**red** dotted).

6. FURTHER INVESTIGATION

We have provided a convex methodology for structured $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller design and a procedure to gain computational efficiency for spatially invariant systems and problems with similar forms. Ongoing work will focus on deriving a bound on the error between a general linear system and the system corresponding to its symmetric component.

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