2014 American Control Conference (ACC)
June 4-6, 2014. Portland, Oregon, USA

Sparsity-promoting optimal control of spatially-invariant systems

David M. Zoltowski, Neil Dhingra, Fu Lin, and Mihailo R. Jovanović

***Abstract*— We study the optimal design of sparse and block sparse feedback gains for spatially-invariant systems on a circle. For this class of systems, the state-space matrices are jointly diagonalizable via the discrete Fourier transform. We exploit this structure to develop an ADMM-based algorithm that significantly reduces the computational complexity relative to standard approaches. Specifically, the complexity of the developed algorithm scales linearly with the number of subsystems. This is in contrast to a cubic scaling when circulant structure is not exploited. Two examples are provided to illustrate the effectiveness of the developed approach.**

***Index Terms*— Alternating direction method of multipliers, Fourier transform, $\mathcal{H}_2$ norm, sparsity-promoting optimal control, spatially-invariant systems, structured feedback control.**

I. Introduction

Optimal design of sparse feedback gains has recently received considerable attention [1]–[8]. Research efforts have focused on developing efficient algorithms to identify controller structures that strike a balance between quadratic performance of distributed systems and sparsity of controllers.

In [1] optimal structured feedback gains have been sought via the augmented Lagrangian approach. In [2], alternating direction method of multipliers (ADMM) was employed for the design of sparse and block sparse feedback gains. The developed algorithm provides a sequence of feedback gains that traces the optimal trade-off curve between the closed-loop $\mathcal{H}_2$ performance and sparsity of the feedback matrix. In [3]–[5], it was shown that for systems with single-integrator dynamics and consensus networks, the sparsity-promoting optimal control problem can be cast as a semidefinite program. Similar results also hold for a class of synchronization networks [6]. In [7], an LMI-based approach was used to design structured dynamic output feedback controllers subject to a given $\mathcal{H}_\infty$ performance. In [8], a convex characterization was provided for an optimal design of row-sparse feedback gains.

In this paper, we consider the design of sparse and block sparse feedback gains for spatially-invariant systems on a circle. This class of systems arises in a variety of applications including formation of vehicles [9], [10], satellite formation flying [11], and cyclic pursuit [12]. For this class of distributed systems, the state-space representation matrices can be jointly diagonalized via the discrete Fourier transform [13]. We exploit this structure to significantly reduce the computational complexity of our algorithm. Specifically, for distributed systems with N subsystems, the complexity of the developed algorithm scales linearly with N. This is in contrast to a cubic scaling with N when the circulant structure is not exploited.

Our work builds on the sparsity-promoting framework developed in [2], [3]. In particular, we augment the objective function with a sparsity-promoting regularizer and employ the ADMM algorithm to achieve an optimal tradeoff between performance and sparsity. In contrast to [2], we focus on a particular class of systems that are governed by spatially-invariant dynamics over a circle. By confining our attention to the class of circulant and block circulant feedback gain matrices, we exploit structure of this problem to gain computational efficiency.

The paper is organized as follows. Section II formulates the sparsity-promoting optimal control problem. Section III describes the ADMM algorithm used to design optimal controllers for spatially-invariant systems with circulant structure. Section IV extends the design method to systems with block circulant structure. Section V provides examples to illustrate the effectiveness of our approach, and Section VI concludes with a summary of our work.

II. Problem formulation

Consider the state-space representation of a spatially invariant system over a circle [13]

$$H \;:=\; \left\{ \begin{array}{rcl} \dot{x} &=& Ax + B_1 d + B_2 u \\ u &=& -Fx \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^N$ is the state vector, $d \in \mathbb{R}^N$ is the stochastic disturbance, $u \in \mathbb{R}^N$ is the control input, and $\{A, B_1, B_2, F\}$ are $N \times N$ circulant matrices. The generalization to block circulant matrices of appropriate dimensions is presented in Section IV.

We consider the design of a sparse feedback gain matrix F that minimizes the steady-state variance amplification (i.e., the $\mathcal{H}_2$ norm) of the closed-loop system

$$J(F) \;:=\; \lim_{t \to \infty} \mathcal{E}\left(x^T(t)Qx(t) \,+\, u^T(t)Ru(t)\right)$$

where $\mathcal{E}$ is the expectation operator and $Q \succeq 0$ and $R \succ 0$ are also circulant matrices. When (A, B_2) is stabilizable and

Financial support from the National Science Foundation under Award CMMI-09-27720, from the University of Minnesota Initiative for Renewable Energy and the Environment under Early Career Award RC-0014-11, and from NSF Research Experience for Undergraduates is gratefully acknowledged.

D. M. Zoltowski is with the Department of Electrical and Computer Engineering, Michigan State University, East Lansing, MI 48825. N. Dhingra and M. R. Jovanović are with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455. F. Lin is with the Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL 60439. E-mails: zoltow11@msu.edu, dhin0008@umn.edu, fulin@mcs.anl.gov, mihailo@umn.edu.

$(A, Q^{1/2})$ is detectable, the optimal state feedback gain is given by

$$F = R^{-1} B_2^T P$$

where P is the unique positive definite solution of the algebraic Riccati equation

$$A^T P + PA + Q - PB_2 R^{-1} B_2^T P = 0.$$

The feedback matrix F is in general a dense matrix. In [2], the authors employed sparsity-promoting penalty functions to design sparse feedback gains. The design process consists of two steps. The first step aims at identifying sparsity structures that strike a balance between the $\mathcal{H}_2$ performance and feedback gain sparsity. The second step aims at finding the optimal feedback gain subject to the identified sparsity structure.

In this paper, we examine the same design problem in the context of spatially-invariant systems on a circle. For this class of systems, we exploit the resulting circulant structure and significantly improve computational efficiency of the algorithms developed in [2].

A. Diagonalization via Fourier transform

Let Φ be the discrete Fourier matrix of size $N \times N$. By introducing the change of variables $\hat{x} := \Phi x$, $\hat{u} := \Phi u$, and $\hat{d} := \Phi d$, the system H can be written as

$$\hat{H} := \begin{cases} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_1 \hat{d} + \hat{B}_2 \hat{u} \\ \hat{u} &= -\hat{F}\hat{x} \end{cases}$$

where $\hat{F}$ is a diagonal matrix determined by

$$\hat{F} := \Phi F \Phi^*.$$

Similarly, $\{\hat{A}, \hat{B}_1, \hat{B}_2\}$ are diagonal matrices obtained using the discrete Fourier transform. The $\mathcal{H}_2$ norm of the closed-loop system is determined by [13]

$$J(\hat{F}) = \begin{cases} \text{trace}\left(\hat{B}_1^* \hat{P} \hat{B}_1\right), & \hat{F} \text{ stabilizing} \\ \infty, & \text{otherwise} \end{cases}$$

where $\hat{P}$ is the closed-loop observability Gramian,

$$(\hat{A} - \hat{B}_2\hat{F})^* \hat{P} + \hat{P}(\hat{A} - \hat{B}_2 \hat{F}) = -(\hat{Q} + \hat{F}^* \hat{R} \hat{F}). \quad (2)$$

Since all matrices in (2) are diagonal matrices of sizes $N \times N$, the Lyapunov equation (2) can be solved with $O(N)$ operations. This is in contrast to $O(N^3)$ operations required to solve Lyapunov equations with dense matrices.

Furthermore, J can be expressed as a function of the diagonal elements $\{\hat{F}_i\}_{i=1}^N$ of the diagonal matrix $\hat{F}$

$$J(\hat{F}) = \sum_{i=1}^{N} J_i(\hat{F}_i) = \sum_{i=1}^{N} \hat{B}_{1i}^T \hat{P}_i \hat{B}_{1i} \quad (3)$$

where the ith diagonal element $\hat{P}_i$ of $\hat{P}$ is determined by the solution to the (scalar) Lyapunov equation

$$(\hat{A}_i - \hat{B}_{2i}\hat{F}_i)^* \hat{P}_i + \hat{P}_i(\hat{A}_i - \hat{B}_{2i}\hat{F}_i) = -(\hat{Q}_i + \hat{F}_i^* \hat{R}_i \hat{F}_i). \quad (4)$$

B. Design of sparse feedback gains

Sparse feedback gains have been identified in [2] by augmenting the $\mathcal{H}_2$ norm with the weighted ℓ_1 norm of the feedback gain matrix

$$g(F) := \sum_{i,j} W_{ij} |F_{ij}|$$

with $W_{ij} > 0$. The weighted ℓ_1 norm was originally used to promote sparsity in signal recovery [14].

The sparsity-promoting optimal control problem is thus given by

$$\begin{array}{ll} \text{minimize} & J(\hat{F}) + \gamma\, g(F) \\ \text{subject to} & \hat{F} = \Phi F \Phi^* \end{array} \quad (5)$$

where $\gamma > 0$ is a sparsity-promoting parameter.

After a sparsity structure $\mathcal{S}$ has been identified from the solution to (5), an optimal feedback gain is obtained by optimizing the controller over the structural constraint set $\mathcal{S}$,

$$\begin{array}{ll} \text{minimize} & J(\hat{F}) \\ \text{subject to} & \hat{F} = \Phi F \Phi^*, \quad F \in \mathcal{S}. \end{array} \quad (6)$$

III. Exploiting structure via ADMM

A. Alternating direction method of multipliers

The sparsity-promoting control problem (5) is challenging because it is in general a non-convex and non-smooth optimization problem [2]. However, $J(\hat{F})$ is a smooth function of the diagonal matrix $\hat{F}$, and $g(F)$ is a convex and separable function in elements of the feedback gain F. By using the alternating direction method of multipliers (ADMM), it is possible to take advantage of smoothness of J and convexity of g. This is achieved by splitting the optimization into two subproblems.

We form the augmented Lagrangian function of the constrained problem (5)

$$\mathcal{L}_\rho(\hat{F}, F, \Lambda) = J(\hat{F}) + \gamma g(F) + \text{trace}\left(\Lambda^T (\hat{F} - \Phi F \Phi^*)\right) + \frac{\rho}{2} \|\hat{F} - \Phi F \Phi^*\|_F^2$$

where Λ is the dual variable, ρ is a positive scalar parameter, and $\|\cdot\|_F^2$ is the Frobenius norm. The constrained optimization problem (5) is solved using the following sequence of iterations

$$\hat{F}^{k+1} := \arg\min_{\hat{F}} \mathcal{L}_\rho(\hat{F}, F^k, \Lambda^k) \quad (7a)$$

$$F^{k+1} := \arg\min_{F} \mathcal{L}_\rho(\hat{F}^{k+1}, F, \Lambda^k) \quad (7b)$$

$$\Lambda^{k+1} := \Lambda^k + \rho(\hat{F}^{k+1} - \Phi F^{k+1} \Phi^*) \quad (7c)$$

until the primal and dual residues are sufficiently small [15]

$$\|\hat{F}^k - \Phi F^k \Phi^*\|_F \le \epsilon_{\text{prim}}, \quad \|F^{k+1} - F^k\|_F \le \epsilon_{\text{dual}}.$$

We next show that both the $\hat{F}$-minimization step (7a) and the F-minimization step (7b) can be decomposed into a sequence of optimization problems with scalar variables. As a consequence, their solutions can be computed efficiently.

In particular, the computational complexity for both subproblems grows linearly with N.

B. $\hat{F}$-minimization step (7a)

Using completion of squares, it can be shown that the minimization of the augmented Lagrangian over $\hat{F}$ is equivalent to

$$\text{minimize} \quad J(\hat{F}) \,+\, (\rho/2)\|\hat{F} - \hat{U}^k\|_F^2 \tag{8}$$

where

$$\hat{U}^k \,=\, \Phi F^k \Phi^* \,-\, (1/\rho)\Lambda^k.$$

Here, the optimization variable is $\hat{F}$. Since F^k is a circulant matrix and since Λ^k is a diagonal matrix by construction (cf. (7c)), we conclude that $\hat{U}^k$ is a diagonal matrix. Therefore, problem (8) can be decomposed into N optimization problems with respect to the diagonal elements $\hat{F}_i$

$$\text{minimize} \quad \sum_{i=1}^{N} \Big(J_i(\hat{F}_i) \,+\, (\rho/2)(\hat{F}_i - \hat{U}_i^k)^2 \Big) \tag{9}$$

where $J_i(\hat{F}_i)$ is given by (3). For each diagonal element $\hat{F}_i$, the scalar problem (9) can be solved analytically. Specifically, setting the derivative of the objective function in (9) to zero yields a set of N cubic equations for each $\hat{F}_i$. The roots of these cubic equations can be computed efficiently.

C. F-minimization step (7b)

By completing the squares with respect to F, minimization of the augmented Lagrangian over F amounts to

$$\text{minimize} \quad \gamma g(F) \,+\, (\rho/2)\|F - V^k\|_F^2 \tag{10}$$

where

$$V^k \,=\, \hat{F}^{k+1} \,+\, (1/\rho)\Lambda^k.$$

Here, the optimization variable is F. When g is the weighted ℓ_1 norm

$$g(F) \,=\, \sum_{i,j} W_{ij}|F_{ij}|$$

with $W_{ij} > 0$, the solution to (10) is determined by the soft-thresholding operator [2]

$$F_{ij}^{k+1} = \begin{cases} (1 - \alpha/|V_{ij}|)V_{ij}, & |V_{ij}| > \alpha \\ 0, & |V_{ij}| \leq \alpha \end{cases}$$

where $\alpha = (\gamma/\rho)W_{ij}$.

D. Structured optimal control problem

The solution to (5) yields feedback gain F belonging to the identified sparsity structure $\mathcal{S}$. We next minimize the $\mathcal{H}_2$ norm over a fixed sparsity structure $F \in \mathcal{S}$. To this end, we use the indicator function

$$\phi(F) \,:=\, \begin{cases} 0, & \text{if } F \in \mathcal{S} \\ +\infty, & \text{otherwise} \end{cases}$$

to rewrite the structured $\mathcal{H}_2$ problem (6) as

$$\begin{array}{ll} \text{minimize} & J(\hat{F}) \,+\, \phi(F) \\ \text{subject to} & \hat{F} \,-\, \Phi F \Phi^* \,=\, 0. \end{array} \tag{11}$$

Note that (11) is similar to (5) except for the difference that the sparsity-promoting function g is replaced by the indicator function ϕ.

We use the ADMM algorithm (7) in which the $\hat{F}$-minimization step (7a) is the same as described in Section III-B. On the other hand, the F-minimization step (7b) amounts to

$$\begin{array}{ll} \text{minimize} & \dfrac{\rho}{2}\|F \,-\, V^k\|_F^2 \\ \text{subject to} & F \,\in\, \mathcal{S}. \end{array}$$

The solution to this problem is a projection of V^k on the sparsity structure $\mathcal{S}$. Specifically,

$$F^{k+1} \,=\, V^k \circ I_{\mathcal{S}}$$

where $\circ$ is the elementwise product of two matrices and $I_{\mathcal{S}}$ is the structural identity,

$$I_{\mathcal{S}ij} \,=\, \begin{cases} 1, & \text{if } F_{ij} \text{ is a free variable} \\ 0, & \text{otherwise.} \end{cases}$$

IV. Block circulant systems

We next turn to the design of block sparse feedback gains for spatially invariant systems with block circulant structure. While the design procedure is similar to the one described in Section III, we note the following differences:

1) Instead of being a diagonal matrix, the discrete Fourier transform brings the block circulant feedback gain F to a block diagonal matrix $\hat{F}$.
2) Instead of having analytical solutions for the $\hat{F}$-minimization step (7a), we use an iterative scheme to compute an optimal solution.
3) Finally, instead of using the elementwise soft-thresholding operator, the block-wise counterpart is used in the F-minimization step (7b).

In what follows, we provide details for each of these three items.

A. Block diagonalization via Fourier transform

Let the spatially invariant system be composed of N subsystems with each subsystem having n states. Then the state vector $x \in \mathbb{R}^{nN}$ is composed of a group of N subvectors $x_i \in \mathbb{R}^n$ which denotes the state of the ith subsystem. The matrix $A \in \mathbb{R}^{nN \times nN}$ is block circulant with blocks of the size $n \times n$. For example, when $N = 3$ we have

$$A \,=\, \begin{bmatrix} A_0 & A_1 & A_{-1} \\ A_{-1} & A_0 & A_1 \\ A_1 & A_{-1} & A_0 \end{bmatrix}$$

where the blocks $\{A_0,\, A_{-1},\, A_1\} \in \mathbb{R}^{n\times n}$ can be arbitrary matrices. Similarly, B_1 and B_2 are block circulant with blocks of the sizes $n \times q$ and $n \times m$, respectively, where q is the number of disturbances per subsystem and m is the number of control inputs per subsystem.

Block circulant matrices are block diagonalizable by appropriate discrete Fourier transform [13]. Let the block

Fourier matrix be denoted as

$$\Phi_a \,:=\, \Phi \otimes I_a$$

where I_a is the $a \times a$ identity matrix and Φ is the $N \times N$ discrete Fourier matrix. By introducing the change of variables $\hat{x} := \Phi_r x$, $\hat{u} := \Phi_m u$, and $\hat{d} := \Phi_q d$, system (1) can be written as

$$\hat{H} \,:=\, \left\{ \begin{array}{rcl} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}_1\hat{d} + \hat{B}_2\hat{u} \\ u & = & -\hat{F}\hat{x}, \end{array} \right.$$

where block diagonal matrices $\{\hat{A},\, \hat{B}_1,\, \hat{B}_2,\, \hat{F}\}$ are given by

$$\begin{array}{rclrcl} \hat{A} & := & \Phi_n A \Phi_n^*, & \hat{B}_1 & := & \Phi_n B_1 \Phi_q^* \\ \hat{B}_2 & := & \Phi_n B_2 \Phi_m^*, & \hat{F} & := & \Phi_m F \Phi_n^*. \end{array}$$

Similar to the scalar circulant case, the $\mathcal{H}_2$ norm J is given by (3) where $\hat{P}_i$ is the ith diagonal block of $\hat{P}$ determined from the Lyapunov equation (4).

B. *Solving the $\hat{F}$-minimization problem* (7a)

For block circulant matrices, the $\hat{F}$-minimization step (7a) no longer has a simple analytical solution as in the scalar circulant case. This is because setting the derivative of the objective function in (7a) results in a set of coupled matrix equations [2].

Following [2], we use the Anderson-Moore method to solve problem (7a). This is an iterative scheme that solves two Lyapunov and one Sylvester equation at each iteration. For a fixed $\hat{F}_i$, two Lyapunov equations are first solved

$$\begin{array}{rcl} (\hat{A}_i - \hat{B}_{2i}\hat{F}_i)\hat{L}_i + \hat{L}_i(\hat{A}_i - \hat{B}_{2i}\hat{F}_i)^* & = & -\hat{B}_{1i}\hat{B}_{1i}^* \\ (\hat{A}_i - \hat{B}_{2i}\hat{F}_i)^*\hat{P}_i + \hat{P}_i(\hat{A}_i - \hat{B}_{2i}\hat{F}_i) & = & -(\hat{Q}_i + \hat{F}_i^*\hat{R}_i\hat{F}_i) \end{array}$$

for $\hat{L}_i$ and $\hat{P}_i$, and then the Sylvester equation

$$2(\hat{R}_i\hat{F}_i - \hat{B}_{2i}^*\hat{P}_i)\hat{L}_i \,+\, \rho(\hat{F}_i - \hat{U}_i^k) \,=\, 0$$

is solved for $\hat{F}_i$. It can be shown that these iterations move $\hat{F}_i$ in a descent direction [2]. Thus, in conjunction with appropriate line-search, the Anderson-Moore method converges to a solution to (8).

Without exploiting the block circulant structure in (1), the complexity of solving each step of the Anderson-Moore method scales as $O(n^3N^3)$. Here, N is the number of subsystems and n is the number of states of each subsystem. By exploiting the block-diagonal structure of $\hat{H}$, we perform the Anderson-Moore method on each *block* in (9), so the complexity of each step scales as $O(n^3N)$. Note that the computational complexity scales linearly with respect to the number of subsystems N.

C. *Solving the F-minimization problem* (7b)

The appropriate sparsity-promoting function g for block sparsity is the weighted sum-of-norms [15]

$$g(F) \,=\, \sum_{i,j} W_{ij}\|F_{ij}\|_F.$$

For each block F_{ij}, the solution to (7b) is given by the block soft-thresholding operator [2]

$$F_{ij}^{k+1} = \begin{cases} (1-\alpha/\|V_{ij}\|_F)V_{ij}, & \|V_{ij}\|_F > \alpha \\ 0, & \|V_{ij}\|_F \leq \alpha. \end{cases}$$

V. Examples

In this section, we provide two examples to demonstrate the effectiveness of the developed ADMM-based algorithm. In both examples, we show that the computational complexity scales as a linear function of the number of subsystems.

A. *Circulant structure*

Consider a first-order system with $N = 5$ nodes distributed over a circle. The dynamics of the system are determined by

$$A \,=\, \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}$$

and B_1, B_2, Q, R are identity matrices of the size $N \times N$. The centralized gain is a circulant matrix

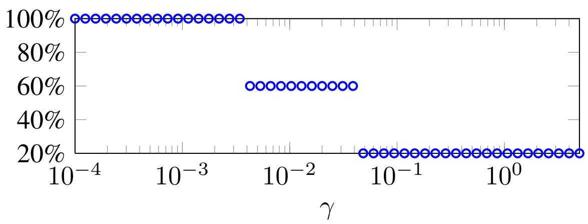
$$F_c \,=\, \begin{bmatrix} 0.3838 & 0.1961 & 0.1120 & 0.1120 & 0.1961 \\ 0.1961 & 0.3838 & 0.1961 & 0.1120 & 0.1120 \\ 0.1120 & 0.1961 & 0.3838 & 0.1961 & 0.1120 \\ 0.1120 & 0.1120 & 0.1961 & 0.3838 & 0.1961 \\ 0.1961 & 0.1120 & 0.1120 & 0.1961 & 0.3838 \end{bmatrix}.$$

For this example, γ is swept logarithmically from 0.001 to 5. The difference between the initial centralized feedback gain and the sparse feedback gain F_{sp} corresponding to $\gamma = 5$ is examined. The sparse gain is a diagonal matrix

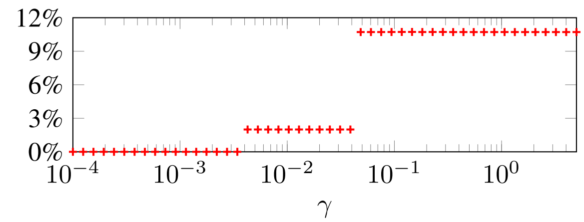
$$F_{sp} = \begin{bmatrix} 0.6848 & 0 & 0 & 0 & 0 \\ 0 & 0.6848 & 0 & 0 & 0 \\ 0 & 0 & 0.6848 & 0 & 0 \\ 0 & 0 & 0 & 0.6848 & 0 \\ 0 & 0 & 0 & 0 & 0.6848 \end{bmatrix}.$$

The level of sparsity and corresponding performance of the optimal feedback gain F are shown in Fig. 1 for different values of γ.

We next consider the computational complexity of the ADMM algorithm with respect to the number of subsystems N. As shown in Fig. 2a, we observe a linear scaling of computational time with respect to N. In Fig. 2b, the scaling of our algorithm for spatially-invariant systems is compared to the algorithm developed in [2] without exploiting circulant structure.

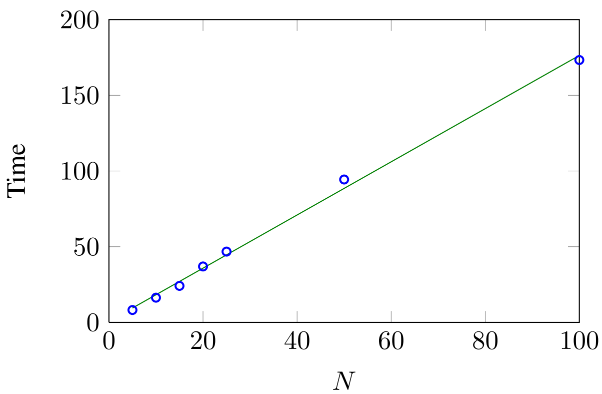


(a) $\mathbf{card}(F)/\mathbf{card}(F_c)$

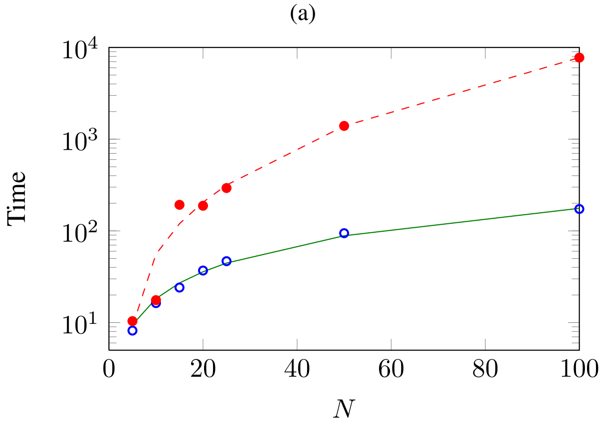


(b) $(J(F) - J(F_c))/J(F_c)$

Fig. 1: (a) The number of nonzero elements of F compared to that of the centralized gain F_c. (b) The $\mathcal{H}_2$ performance deteriorates gracefully with increase in γ.



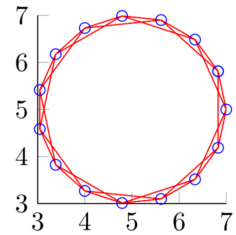
(a)



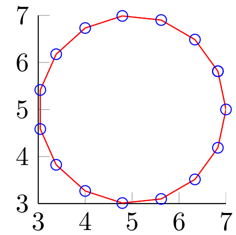
(b)

Fig. 2: (a) The linear scaling of computational time with the number of subsystems N. (b) The comparison between the linear scaling (o) for the algorithm developed in this paper and the cubic scaling (•) using the algorithm developed in [2].

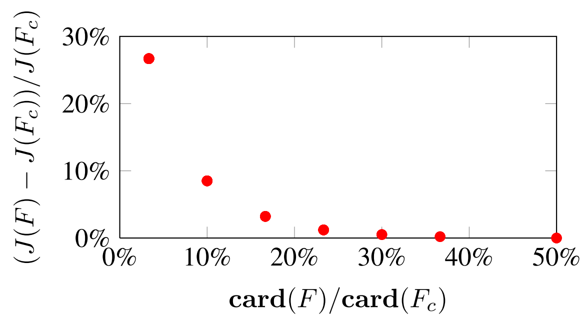
B. Block circulant structure



(a) $\gamma = 0.1389$



(b) $\gamma = 0.9103$



(c)

Fig. 3: (a)-(b) The communication graphs of controllers obtained for different values of γ. (c) The optimal trade-off curve between the performance degradation and sparsity level relative to F_c.

We next provide an example that illustrates the performance of our algorithm on a block circulant system. Let $N = 15$ subsystems be evenly distributed on a circle and let each subsystem be an unstable second order system. The dynamics of a subsystem are coupled with the dynamics of other subsystems through an exponentially decaying function of the Euclidean distance $\alpha(i,j)$ between them [16]

$$\begin{bmatrix} \dot{p}_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} p_i \\ v_i \end{bmatrix} + \sum_{j \neq i} e^{\alpha(i,j)} \begin{bmatrix} p_j \\ v_j \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (d_i + u_i).$$

The state and control weights Q and R are identity matrices. The values of γ are chosen at 48 logarithmically spaced points between 0.01 and 68.66.

As γ increases, the communication graphs become sparser and sparser; see Fig. 3 for two different identified communication graphs. The trade-off between optimal performance and sparsity is shown in Fig. 3c.

Similar to the case for scalar systems, the ADMM algorithm scales linearly with the number of subsystems N, as opposed to the cubic scaling for the method that does not exploit circulant structure; see Fig. 4. For this case, the method developed in [2] is observed to be faster for $N < 40$. For large enough number of subsystems, the overhead of performing additional Fourier Transforms becomes insignificant.

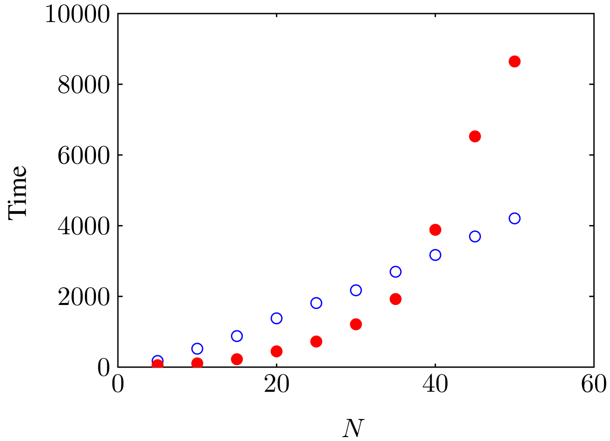


Fig. 4: The computation time vs the number of subsystems N for the ADMM algorithms that exploit the block circulant structure (○) and that does not exploit structure (●).

VI. CONCLUDING REMARKS

We consider the design of sparse and block sparse feedback gains for spatially invariant systems on a circle. By exploiting the circulant structure, we significantly improve the computational efficiency of the ADMM algorithm. In particular, the algorithmic complexity scales linearly with the number of subsystems as opposed to a cubic dependence when the structure is not exploited. The ADMM algorithm has been implemented in MATLAB.

The developed method is most efficient in applications with a large number of subsystems where each subsystem has a small number of states. Since both these features are commonly encountered in multi-agent systems [17], consensus and synchronization networks, the developed approach may find use in a host of emerging applications.

REFERENCES

[1] F. Lin, M. Fardad, and M. R. Jovanović, "Augmented Lagrangian approach to design of structured optimal state feedback gains," *IEEE Trans. Automat. Control*, vol. 56, no. 12, pp. 2923–2929, December 2011.

[2] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Automat. Control*, vol. 58, no. 9, pp. 2426–2431, 2013.

[3] M. Fardad, F. Lin, and M. R. Jovanović, "Sparsity-promoting optimal control for a class of distributed systems," in *Proceedings of the 2011 American Control Conference*, San Francisco, CA, 2011, pp. 2050–2055.

[4] F. Lin, M. Fardad, and M. R. Jovanović, "Identification of sparse communication graphs in consensus networks," in *Proceedings of the 50th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, 2012, pp. 85–89.

[5] D. Zelazo, S. Schuler, and F. Allgöwer, "Performance and design of cycles in consensus networks," *Syst. Control Lett.*, vol. 62, no. 1, pp. 85–96, 2013.

[6] M. Fardad, F. Lin, and M. R. Jovanović, "Design of optimal sparse interconnection graphs for synchronization of oscillator networks," *IEEE Trans. Automat. Control*, 2014, doi:10.1109/TAC.2014.2301577.

[7] S. Schuler, P. Li, J. Lam, and F. Allgöwer, "Design of structured dynamic output-feedback controllers for interconnected systems," *International Journal of Control*, vol. 84, no. 12, pp. 2081–2091, 2011.

[8] B. Polyak, M. Khlebnikov, and P. Shcherbakov, "An LMI approach to structured sparse feedback design in linear control systems," in *Proceedings of the 2013 European Control Conference*, 2013, pp. 833–838.

[9] M. R. Jovanović and B. Bamieh, "On the ill-posedness of certain vehicular platoon control problems," *IEEE Trans. Automat. Control*, vol. 50, no. 9, pp. 1307–1321, September 2005.

[10] B. Bamieh, M. R. Jovanović, P. Mitra, and S. Patterson, "Coherence in large-scale networks: dimension dependent limitations of local feedback," *IEEE Trans. Automat. Control*, vol. 57, no. 9, pp. 2235–2249, September 2012.

[11] P. Massioni and M. Verhaegen, "Distributed control for identical dynamically coupled systems: a decomposition approach," *IEEE Trans. Automat. Control*, vol. 54, no. 1, pp. 124–135, 2009.

[12] J. A. Marshall, M. E. Broucke, and B. A. Francis, "Formations of vehicles in cyclic pursuit," *IEEE Trans. Automat. Control*, vol. 49, no. 11, pp. 1963–1974, 2004.

[13] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems," *IEEE Trans. Automat. Control*, vol. 47, no. 7, pp. 1091–1107, 2002.

[14] E. J. Candès, M. B. Wakin, and S. P. Boyd, "Enhancing sparsity by reweighted ℓ_1 minimization," *Journal of Fourier Analysis and Applications*, vol. 14, pp. 877–905, 2008.

[15] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends in Machine Learning*, vol. 3, no. 1, pp. 1–124, 2011.

[16] N. Motee and A. Jadbabaie, "Optimal control of spatially distributed systems," *IEEE Trans. Automat. Control*, vol. 53, no. 7, pp. 1616–1629, 2008.

[17] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.